INDEX-DOUBLING IN SEQUENCES BY AITKEN ACCELERATION

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ABSTRACT

Let (x_n) denote a sequence which is linearly convergent to a limit x, and whose second differences are all nonzero. For a large class of such sequences (x_n), the associated sequence (x_n^*) defined by

$$x_n^* = \frac{x_{n+1}x_{n-1} - x_n^2}{x_{n+1} - 2x_n + x_{n-1}}$$

converges to x faster than (x_n). The derivation of this associated sequence is called Aitken acceleration. In a paper published in 1984 I showed that, with $x_n = F_{n+1}/F_n$,

$$\frac{x_{n+r}x_{n-r} - x_n^2}{x_{n+r} - 2x_n + x_{n-r}} = x_{2n}$$

for $1 \leq r < n$ so that, in particular, $x_n^* = x_{2n}$. Thus, we have an associated sequence that is a subsequence of the original sequence. A number of authors have followed up this result and in the present paper I summarize the progress made on this topic to date and present some new results.

1. INTRODUCTION

Let (x_n) denote a sequence of real numbers that converges to the limit x, and let

$$x_n^* = \frac{x_{n+1}x_{n-1} - x_n^2}{x_{n+1} - 2x_n + x_{n-1}}, \tag{1}$$

assuming that the denominator on the right side of (1) is nonzero. The derivation of the associated sequence (x_n^*) is called Aitken acceleration, after A. C. Aitken (1895–1967). Observe that x_n^* immediately yields the limit x when

$$x_n - x = a\rho^n,$$

where a and ρ are real with $0 < |\rho| < 1$.

Given a sequence (x_n) with limit x such that

$$\frac{x_{n+1} - x}{x_n - x} = \rho + \epsilon_n,$$

where $0 < |\rho| < 1$ and $\epsilon_n \to 0$ as $n \to \infty$, then (see Henrici [5]) the sequence (x_n^*) is defined for n sufficiently large and converges to x faster than the original sequence (x_n), in the sense that

$$\lim_{n\to\infty} \frac{x_n^* - x}{x_n - x} = 0.$$

Aitken acceleration was routinely used to speed up convergence of a sequence in the latter part of the period when extensive numerical work was carried out on desk machines.

Circa 1965, I found by chance a sequence (x_n) whose accelerated sequence (x_n^*) is a *subsequence* of (x_n). In a discussion of simple iterative methods in an introductory course on numerical analysis, I considered the test equation $x^2 - x - 1 = 0$ and found the positive root by using the iterative process

$$x_{n+1} = 1 + \frac{1}{x_n}, \quad n \geq 1,$$

with $x_1 = 1$. In this case

$$x_n = \frac{F_{n+1}}{F_n}, \quad n \geq 1, \tag{2}$$

where (F_n) is the Fibonacci sequence. The sequence (x_n) converges to the golden ratio, $\frac{1}{2}\left(\sqrt{5}+1\right)$, the positive root of $x^2 - x - 1$, and we can show (see [9]) that

$$\frac{x_{n+r}x_{n-r} - x_n^2}{x_{n+r} - 2x_n + x_{n-r}} = \frac{F_{2n+1}}{F_{2n}} = x_{2n} \tag{3}$$

for $n > r > 0$. In particular, on putting $r = 1$ in (3), we obtain

$$x_n^* = x_{2n}. \tag{4}$$

If we use Aitken acceleration on (x_n^*) to give a sequence (x_n^{**}), we see from (4) and (3) that

$$x_n^{**} = x_{2n}^* = x_{4n}.$$

We can continue by accelerating the sequence (x_n^{**}), and so on. With each acceleration we double the suffix in the sequence defined by (2). These ideas have been pursued by others, including Brezinski and Lembarki [4], Jamieson [6, 7], and Alexander [3].

It is interesting to note what A. C. Aitken himself had to say about the Fibonacci numbers. The following quotation is the first sentence of a letter that Aitken wrote to the biologist D'Arcy Thompson (1860–1948) on 20 December 1938:

> *Any mention of the Fibonacci numbers is always sure to draw me. All the romance of continued fractions, linear recurrence relations, surd approximations to integers and the rest lies in them; and they are a source of endless curiosity.*

For biographical information about Aitken, see [1, 2, 10].

2. GENERALIZED FIBONACCI NUMBERS

We can generalize the Fibonacci numbers by writing

$$U_{n+1} = aU_n - bU_{n-1}, \tag{5}$$

with $U_1 = 1$ and $U_2 = a$, where a and b are nonzero real numbers. Let α and β denote the roots of the characteristic equation

$$x^2 - ax + b = 0, \tag{6}$$

so that

$$\alpha + \beta = a \quad \text{and} \quad \alpha\beta = b. \tag{7}$$

Then it is easily verified that

$$U_n = \begin{cases} \frac{\alpha^n - \beta^n}{\alpha - \beta}, & \alpha \neq \beta, \\ n\alpha^{n-1}, & \alpha = \beta. \end{cases} \tag{8}$$

The Fibonacci sequence is recovered by choosing $a = -b = 1$. Note that U_n is defined for *all integers* n, and that $U_0 = 0$. If we wish to define $x_n = U_{n+1}/U_n$, we need to know when U_n is zero. (For example, if we choose $a = 1$ and $b = \frac{1}{3}$, we find that $U_6 = 0$.) It is shown in [8] that, for $n > 0$, $U_n = 0$ only if the roots of the characteristic equation $x^2 - ax + b = 0$ are complex and

$$\cos^{-1}\left(\frac{1}{2}a/\sqrt{b}\right) = \frac{k\pi}{n},$$

where k is an integer. (Note that b is positive when the roots of the characteristic equation are complex.) Thus, when $n > 1$, U_n is zero only when $a^2 < 4b$ and $\frac{1}{2}a/\sqrt{b}$ is a zero of the Chebyshev polynomial of the second kind of degree $n-1$. The following theorem (see [8]) generalizes (3).

Theorem 2.1: Let $x_n = U_{n+1}/U_n$. Then for those values of x_{n-r}, x_n, and x_{n+r} that are defined, we have

$$\frac{x_{n+r}x_{n-r} - x_n^2}{x_{n+r} - 2x_n + x_{n-r}} = \frac{U_{2n+1}}{U_{2n}} = x_{2n} \tag{9}$$

for all $r \neq 0$.

In what follows we will also require the generalized Lucas sequence (V_n), defined by

$$V_{n+1} = aV_n - bV_{n-1}, \tag{10}$$

with $V_0 = 2$ and $V_1 = a$, where a and b are nonzero real numbers. Then it is easily verified that

$$V_n = \alpha^n + \beta^n, \tag{11}$$

for all integers n, where α and β are defined by (7). We recover the Lucas sequence from (V_n) by putting $a = -b = 1$. If we choose $a = b = 3$, we find that $V_3 = 0$. Clearly V_n can only be zero if α and β are a complex conjugate pair. Then $b = \alpha\beta = \alpha\bar{\alpha}$ must be positive and we can write

$$\alpha = b^{1/2}e^{i\theta} \quad \text{and} \quad \beta = b^{1/2}e^{-i\theta}. \tag{12}$$

We see from (7) that

$$a = 2b^{1/2}\cos\theta,$$

and we obtain from (11) that

$$V_n = 2b^{n/2}\cos n\theta.$$

The condition for V_n to be zero is that $\cos n\theta = 0$, which yields the condition

$$\cos^{-1}\left(\frac{1}{2}a/\sqrt{b}\right) = \frac{(2k-1)\pi}{2n}, \tag{13}$$

where k is an integer. Thus, when n is positive, V_n is zero only when $a^2 < 4b$ and $\frac{1}{2}a/\sqrt{b}$ is a zero of the Chebyshev polynomial of degree n.

The generalized Fibonacci numbers and generalized Lucas numbers are both special cases of the sequence (W_n) defined by

$$W_{n+1} = aW_n - bW_{n-1}, \tag{14}$$

where W_0 and W_1 are arbitrary real numbers. We can verify by induction that

$$W_n = \left(W_1 - \frac{1}{2}aW_0\right)U_n + \frac{1}{2}W_0V_n. \tag{15}$$

3. FURTHER RESULTS

The following lemma is easily verified.

Lemma 3.1: Given any sequence (x_n), let us define

$$y_n = \lambda x_n + \mu, \tag{16}$$

where λ and μ are independent of n. Then we have

$$\frac{y_{n+r}y_{n-r} - y_n^2}{y_{n+r} - 2y_n + y_{n-r}} = \lambda\left(\frac{x_{n+r}x_{n-r} - x_n^2}{x_{n+r} - 2x_n + x_{n-r}}\right) + \mu. \tag{17}$$

Before applying Lemma 3.1, we require the following identities, which can be verified by using (8) and (11) or by induction:

$$U_{n+k} = U_kU_{n+1} - bU_{k-1}U_n, \tag{18}$$

$$V_{n+k} = V_k U_{n+1} - bV_{k-1} U_n. \tag{19}$$

If we multiply (18) by $W_1 - \frac{1}{2}aW_0$, multiply (19) by $\frac{1}{2}W_0$, and add, we see from (15) that

$$W_{n+k} = W_k U_{n+1} - bW_{k-1} U_n. \tag{20}$$

Alternatively we can verify (20) directly, using induction. We can now generalize Theorem 2.1.

Theorem 3.1: For any nonzero integer k, let $y_n = W_{n+k}/U_n$. Then for those values of y_{n-r}, y_n, and y_{n+r} that are defined, we have

$$\frac{y_{n+r}y_{n-r} - y_n^2}{y_{n+r} - 2y_n + y_{n-r}} = y_{2n} \tag{21}$$

for all $r \neq 0$. Thus we obtain index-doubling when we apply Aitken acceleration repeatedly to the sequence (y_n).

Proof: Let $x_n = U_{n+1}/U_n$. Then it follows from (20) that

$$y_n = W_k x_n - bW_{k-1}, \tag{22}$$

and we see from Lemma 3.1 and Theorem 2.1 that

$$\frac{y_{n+r}y_{n-r} - y_n^2}{y_{n+r} - 2y_n + y_{n-r}} = W_k \frac{U_{2n+1}}{U_{2n}} - bW_{k-1}.$$

On applying (20) we find that

$$\frac{y_{n+r}y_{n-r} - y_n^2}{y_{n+r} - 2y_n + y_{n-r}} = \frac{W_{2n+k}}{U_{2n}} = y_{2n},$$

and this completes the proof.

Jamieson [6] proved Theorem 3.1 for the case where $a = -b = 1$, $W_0 = 0$, $W_1 = 1$, so that $y_n = F_{n+k}/F_n$.

As another special case of the sequence defined by $y_n = W_{n+k}/U_n$, let us choose $W_0 = 2$, $W_1 = a$, put $k = 0$, and let α and β in (7) be a complex conjugate pair. Then we can write (see (12))

$$\alpha = b^{1/2}e^{i\theta} \quad \text{and} \quad \beta = b^{1/2}e^{-i\theta},$$

where

$$\theta = \cos^{-1}\left(\frac{1}{2}a/\sqrt{b}\right).$$

We thus obtain

$$y_n = \frac{V_n}{U_n} = 2b^{1/2}\sin\theta\,\cot n\theta,$$

and it follows as a corollary of Theorem 3.1 that we obtain index doubling when we apply the Aitken acceleration process repeatedly to the sequence whose nth term is $\cot n\theta$. This result was obtained, using another approach, by Jamieson [7].

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