MACHIN-TYPE FORMULAS EXPRESSING π IN TERMS OF ϕ

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Abstract. In this paper, we prove some formulas for π that are expressed in terms of the powers of the reciprocal of the Golden Ratio ϕ. These formulas depend on Machin-type identities like the following:

$$\pi = 12 \arctan\left(\frac{1}{\phi^3}\right) + 4 \arctan\left(\frac{1}{\phi^5}\right).$$

1. Introduction

In a recent paper [18], we proved several formulas of π which are expressed in terms of the reciprocal of the Golden Ratio ϕ; for example

$$\pi = \frac{5\sqrt{2+\phi}}{2\,\phi} \sum_{n=0}^{\infty} \left(\frac{1}{2\phi}\right)^{5n} \left(\frac{1}{5n+1} + \frac{1}{2\phi^2(5n+2)} - \frac{1}{2^2\phi^3(5n+3)} - \frac{1}{2^3\phi^3(5n+4)}\right) \tag{1.1}$$

These formulas were inspired by the work of Bailey, Borwein and Plouffe (BBP) [6], who proved a family of amazing formulas for π with the aid of the powerful PSLQ algorithm [19]. As an example, they proved that

$$\pi = \sum_{n=0}^{\infty} \frac{1}{16^n} \left(\frac{4}{8n+1} - \frac{2}{8n+4} - \frac{1}{8n+5} - \frac{1}{8n+6}\right). \tag{1.2}$$

For an introduction and generalizations of (1.2), see, e.g., [1, 2, 7]; see also the lucid account in Hijab's book [20]. For a compendium of currently known results of BBP-type formulas, see Bailey's *A Compendium of BBP-Type Formulas for Mathematical Constants*, which is available at `http://crd.lbl.gov/∼dhbailey`.

In this paper, we prove several formulas for π that share similarities with (1.1). they express π in terms of the reciprocal of the Golden Ratio ϕ:

$$\begin{aligned}
\frac{\pi}{4} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{\phi}\right)^{2k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{\phi^3}\right)^{2k+1}, && (1.3)\\
&= 2\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{\phi^2}\right)^{2k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{\phi^6}\right)^{2k+1}, && (1.4)\\
&= 3\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{\phi^3}\right)^{2k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{\phi^5}\right)^{2k+1}. && (1.5)
\end{aligned}$$

The derivations of these formulas (cf. the next section) are analogous to the formula discovered by Machin (1680-1752).

Machin's discovery starts with the following observation:

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right). \tag{1.6}$$

By applying to (1.6) the power series of $\arctan x$, i.e.,

$$\arctan x = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1}, \tag{1.7}$$

we have the Machin's formula for π:

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{5}\right)^{2k+1} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}\left(\frac{1}{239}\right)^{2k+1}. \tag{1.8}$$

Machin's discovery plays a key role in computing the digits of π, see [17]. See also [8, 13]. For generalizations of Machin's formula, see [15, 17]. See also Weisstein's article [23].

It should be noted that there are Machin-Type formulas that are closely related to the Fibonacci numbers. For example, we have

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \arctan\left(\frac{1}{F_{2k+1}}\right).$$

Compare Chapter 42 of Koshy's book [21]. See also Ron Knott's award-winning website at

`http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fib.html`.

We also remark that in an interesting paper [14], Jon Borwein, David Borwein and William Galway explored a class of Machin-type BBP formulas.

Next, we will turn to the proof of (1.3)-(1.5).

2. Proofs of the Main Formulas

To prove (1.3)-(1.5), all we need to do is to establish the following identities:

$$\frac{\pi}{4} = \arctan\left(\frac{1}{\phi}\right) + \arctan\left(\frac{1}{\phi^3}\right), \tag{2.1}$$

$$= 2\arctan\left(\frac{1}{\phi^2}\right) + \arctan\left(\frac{1}{\phi^6}\right), \tag{2.2}$$

$$= 3\arctan\left(\frac{1}{\phi^3}\right) + \arctan\left(\frac{1}{\phi^5}\right). \tag{2.3}$$

By using (1.7), it follows at once that (2.1) leads to (1.3), (2.2) to (1.4) and (2.3) to (1.5). We will say a few words on how these identities were discovered in the next section.

First, we recall two identities that will be useful in our proof. For $n \geq 2$

$$\phi^n = F_n\phi + F_{n-1} \tag{2.4}$$

and for $n \geq 1$,

$$\phi^{-n} = (-1)^{n-1}F_n\phi + (-1)^n F_{n+1}. \tag{2.5}$$

Here, F_n is the nth Fibonacci number. For proofs, see p. 78 of [21] or p. 138 in [12]. The latter is based on a probabilistic approach; see also [9, 10, 11].

Let us define $\gamma = \arctan\left(1/\phi^3\right)$. In order to prove (2.3), we need to show that

$$\tan\left(\frac{\pi}{4} - 3\gamma\right) = \frac{1}{\phi^5}. \tag{2.6}$$

First, we claim that

$$\tan(2\gamma) = \frac{1}{2}. \tag{2.7}$$

Indeed, we have

$$\begin{aligned} \tan 2\gamma &= \frac{2\tan\gamma}{1-\tan^2\gamma} = \frac{2\phi^{-3}}{1-\phi^{-6}} \\ &= \frac{2}{\phi^3 - \phi^{-3}} \\ &= \frac{2}{F_3\phi + F_2 - (F_3\phi - F_4)} = \frac{1}{2}. \end{aligned}$$

Note that we have used (2.4) and (2.5) for $\phi^{\pm 3}$ to derive the third equality.

Next, we show that in a similar manner

$$\tan(3\gamma) = \frac{3+2\phi}{1+4\phi}. \tag{2.8}$$

Indeed, by using (2.7) for $\tan 2\gamma$ we have

$$\tan 3\gamma = \frac{\tan\gamma + \tan 2\gamma}{1 - \tan\gamma\tan 2\gamma} = \frac{2+\phi^3}{2\phi^3 - 1} = \frac{3+2\phi}{1+4\phi}.$$

Note that we have used (2.4) and (2.5) to obtain the last equality.

With (2.8), we can establish (2.6):

$$\tan\left(\frac{\pi}{4} - 3\gamma\right) = \frac{1-\tan 3\gamma}{1+\tan 3\gamma} = \frac{\phi - 1}{3\phi + 2} = \frac{\phi^{-1}}{\phi^4} = \frac{1}{\phi^5}.$$

In the third equality, we have used $\phi - 1 = \phi^{-1}$ (i.e., $n = 1$ in (2.5)) to rewrite the numerator, and $3\phi + 2 = \phi^4$ (i.e., $n = 4$ in (2.4)) to rewrite the denominator. This establishes (2.3) and implies (1.5).

The proofs of the other two equations, namely, (2.1) and (2.2), follow the same pattern and we briefly comment on these proofs.

For (2.1), let us define $\alpha = \arctan\left(1/\phi\right)$. Then,

$$\tan\left(\frac{\pi}{4} - \alpha\right) = \frac{1-\tan\alpha}{1+\tan\alpha} = \frac{\phi-1}{\phi+1} = \frac{\phi^{-1}}{\phi^2} = \frac{1}{\phi^3}.$$

Note that we have used the fact that $\phi - 1 = \phi^{-1}$ (i.e., $n = 1$ in (2.5)) and $\phi^2 = \phi + 1$ (i.e., $n = 2$ in (2.4)). This proves (2.1).

For (2.2), let us define $\beta = \arctan\left(1/\phi^2\right)$. Then, we have

$$\tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2}{\phi^2 - \phi^{-2}} = \frac{2}{2\phi - 1}.$$

Note that we have used (2.4) and (2.5) to obtain the last line. Finally,

$$\tan\left(\frac{\pi}{4} - 2\beta\right) = \frac{1-\tan 2\beta}{1+\tan 2\beta} = \frac{2\phi - 3}{2\phi + 1} = \frac{\phi^{-3}}{\phi^3} = \frac{1}{\phi^6}.$$

This proves (2.2). □

As a by-product, we note that (2.7) implies the following identity

$$\arctan\left(\frac{1}{\phi^3}\right) = \frac{1}{2}\arctan\left(\frac{1}{2}\right).$$

3. Looking Back and Ahead

Originally, identities (2.1) to (2.3) were discovered in a series of numerical experiments using *Mathematica*. The following is an account of how we discovered (2.2).

Motivated by Machin formula (1.6), we first set out to look for an identity of the following form:

$$\frac{\pi}{4} = A \arctan\left(\frac{1}{\phi^2}\right) + B \arctan\left(\frac{1}{\phi^k}\right). \tag{3.1}$$

Our goal was to determine constants A, B and k.

Next, we treated the first term in (3.1) as a first order approximation of $\pi/4$:

$$\frac{\pi}{4} \simeq A \arctan\left(\frac{1}{\phi^2}\right).$$

We compared this approximation with the following numerical result

$$\frac{\pi}{4} \simeq 2.15 \arctan\left(\frac{1}{\phi^2}\right).$$

This motivated us to set $A = 2$ in (3.1):

$$\frac{\pi}{4} = 2 \arctan\left(\frac{1}{\phi^2}\right) + B \arctan\left(\frac{1}{\phi^k}\right). \tag{3.2}$$

We then looked for B and k that would make (3.2) an identity. We performed a series of numerical experiments for this purpose. Precisely, we used *Mathematica* to compute, for a range of k, the following ratio which represents B (cf. (3.2)):

$$\frac{\frac{\pi}{4} - 2\arctan\left(\frac{1}{\phi^2}\right)}{\arctan\left(\frac{1}{\phi^k}\right)}.$$

Gladly, we found that at $k = 6$ (a relatively small k!), *Mathematica* computed the ratio to be

$$1.000000000000000000000000000000$$

(where we requested the software to give the first 30 decimal places). This suggested that we should try to prove or disprove the identity

$$\frac{\pi}{4} \stackrel{?}{=} 2\arctan\left(\frac{1}{\phi^2}\right) + \arctan\left(\frac{1}{\phi^6}\right). \tag{3.3}$$

With joy and gratitude, we found that (3.3) turned out to be an exact identity.

One can generalize the present work by considering recursions with three or more terms. See [23].

We believe that more sophisticated numerical experiments, along the overall theme suggested in [15, 16], may point to more discoveries of this type of identities. See also [3, 4, 5].

Notes Added. Benoit Cloitre, using integer relation methods [15], discovered empirically a beautiful formula that is a higher-order identity of (2.1)-(2.3):

$$\frac{\pi^2}{50} = \sum_{k=0}^{\infty} \left\{ \frac{\phi^2}{(5k+1)^2} - \frac{\phi}{(5k+2)^2} - \frac{\phi^2}{(5k+3)^2} + \frac{\phi^5}{(5k+4)^2} + \frac{2\phi^2}{(5k+5)^2} \right\} \phi^{-5k}. \tag{3.4}$$

This is also one of the SIAM Problems (Problem 06-003, *A Golden Example*, cf. `http://www.siam.org/journals/categories/06-003.php`) proposed and solved elegantly by Jonathan Borwein and Marc Chamberland. It is likely that similar methods can be applied to give a new proof of (2.1)-(2.3). I would like to thank the referee and Benoit Cloitre who brought to my attention such a wonderful identity.

Acknowledgment

I would like to thank Scott Ebbing for his valuable comments on this work. I would like to thank Benoit Cloitre, Ron Knott, Florian Luca and Pante Stanica for their interest in this work. Indeed, quite recently, Luca and Stanica [22] found some wonderful extensions of (2.1)-(2.3). Last but not least, I would like to thank the referee for encouragement and suggestions.

References

[1] V. Adamchik and S. Wagon, *π: A 2000-Year Search Changes Direction*, Mathematica in Education and Research, 1 (1996), 11–19.
[2] V. Adamchik and S. Wagon, *A Simple Formula for Pi*, Amer. Math. Monthly, 104 (1997), 852–855.
[3] D. H. Bailey and J. M. Borwein, *Experimental Mathematics: Recent Developments and Future Outlook*, Mathematics Unlimited – 2001 and Beyond, Bjorn Engquist and Wilfried Schmid, ed., Springer, New York, 2001, 51–66.
[4] D. H. Bailey and J. M. Borwein, *Experimental Mathematics: Examples, Methods and Implications*, Notices Amer. Math. Soc., 52 (2005), 502–514.
[5] D. H. Bailey, J. M. Borwein, V. Kapoor and E. Weisstein, *Ten Problems in Experimental Mathematics*, MAA Monthly, (to appear).
[6] D. H. Bailey, P. B. Borwein and S. Plouffe, *On the Rapid Computation of Various Polylogarithmic Constants*, Math. Comp., 66 (1997), 903–913.
[7] D. H. Bailey and S. Plouffe, *Recognizing Numerical Constants*, The Organic Mathematics Project Proceedings, `http://www.cecm.sfu.ca/organics`,April 12, 1996; hard copy version: Canadian Mathematical Society Conference Proceedings, 20 (1997) 73–88.
[8] P. Beckmann, *The History of π*, 3rd ed., St. Martin's Griffin, New York, 1976.
[9] A. T. Benjamin, C. R. H. Hanusa, and F. E. Su, *Linear Recurrences Through Tilings and Markov Chains*, Utilitas Mathematica, 64 (2003), 3–17.
[10] A. T. Benjamin, G. M. Levin, K. Mahlburg, and J. J. Quinn, *Random Approaches to Fibonacci Identities*, Amer. Math. Monthly, 107 (2000), 511–516.
[11] A. T. Benjamin, J. D. Neer, D. E. Otero, and J. A. Sellers, *A Probabilistic View of Certain Weighted Fibonacci Sums*, The Fibonacci Quarterly, 41 (2003), 360–364.
[12] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count: The Art of Combinatorial Proof*, MAA, Washington, 2003.
[13] D. Blatner, *The Joy of π*, Walker and Company, New York, 1997.
[14] D. Borwein, J. M. Borwein, and W. F. Galway, *Finding and Excluding -ary Machin-Type BBP Formulae*, Canadian J. Math, 56 (2004), 897–925.
[15] J. M. Borwein and D. H. Bailey, *Mathematics by Experiment: Plausible Reasoning in the 21st Century*, A. K. Peters, Massachusetts, 2004.

[16] J. M. Borwein, D. H. Bailey, R. Girgensohn, *Experimentation in Mathematics: Computational Paths to Discovery*, A. K. Peters, Massachusetts, 2004.
[17] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity*, CMS Series of Monographs and Advanced books in Mathematics, John Wiley, New Jersey, 1987.
[18] H. C. Chan, *π in Terms of ϕ*, The Fibonacci Quarterly, 44.2 (2006), 141–144.
[19] H. R. P. Ferguson, D. H. Bailey and S. Arno, *Analysis of PSLQ, An Integer Relation Finding Algorithm*, Mathematics of Computation, 68 (1999), 351–369.
[20] O. Hijab,*Introduction to Calculus and Classical Analysis*, Springer-Verlag, New York, 1997.
[21] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley, New York, 2001.
[22] F. Luca and P. Stanica, *On Machin's Formula with Powers of the Golden Section*, International J. Number Theory, (to appear).
[23] E. W. Weisstein, *Machin-Like Formulas*, From MathWorld–A Wolfram Web Resource, `http://mathworld.wolfram.com/Machin-LikeFormulas.html`.

MSC2000: 11A05

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