A NOTE ON RAMUS' IDENTITY AND ASSOCIATED RECURSION RELATIONS

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ABSTRACT. A system of recursion relations is used to establish Ramus' identity for sums of binomial coefficients in arithmetic progression. As a by-product another recursion relation is obtained, which has also been used to derive Ramus' identity.

1. INTRODUCTION

Consider the sums of binomial coefficients

$$\begin{cases} A_0^{(k)}(n) = \sum_{m=0}^{[n/k]} \binom{n}{km}, \\ \text{and} \quad A_j^{(k)}(n) = \sum_{m=1}^{[(n+j)/k]} \binom{n}{km-j}, \quad j = 1, 2, \ldots, k-1, \quad n \geq 1. \end{cases} \tag{1}$$

In [4] Ramus obtained the following formula for these sums:

$$A_j^{(k)}(n) = \frac{1}{k} \sum_{p=0}^{k-1} \omega_p^j (1 + \omega_p)^n, \quad j = 0, 1, \ldots, k-1, \quad n \geq 1, \tag{2}$$

where the ω_p's are the kth roots of unity:

$$\omega_p = e^{ip2\pi/k}, \quad p = 0, 1, \cdots, k-1. \tag{3}$$

His derivation of this identity is described in [1] as well as in [5].

In [2] Konvalina and Liu gave a different proof of (2). Their proof was based on showing that for each fixed $j = 0, 1, \ldots, k-1$, $A_j^{(k)}(n)$ satisfied the recursion relation

$$A_j^{(k)}(n+k) = \sum_{l=1}^{k-1} (-1)^{l-1} \binom{k}{l} A_j^{(k)}(n+k-l) + \left[1 + (-1)^{k-1}\right] A_j^{(k)}(n), \quad n \geq 1. \tag{4}$$

In this note we want to give another derivation of (2), based also on recursion relations. In our case, in contrast to (4), our recursions are a system of relations satisfied simultaneously by all the $A_j^{(k)}$'s. Namely

$$\begin{cases} A_0^{(k)}(n) = A_0^{(k)}(n-1) + A_1^{(k)}(n-1), \\ A_1^{(k)}(n) = A_1^{(k)}(n-1) + A_2^{(k)}(n-1), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ A_{k-2}^{(k)}(n) = A_{k-2}^{(k)}(n-1) + A_{k-1}^{(k)}(n-1), \\ A_{k-1}^{(k)}(n) = A_{k-1}^{(k)}(n-1) + A_0^{(k)}(n-1), \quad n \geq 2, \end{cases} \tag{5}$$

with the initial conditions

$$A_0^{(k)}(1) = 1, \quad A_1^{(k)}(1) = A_2^{(k)}(1) = \cdots = A_{k-2}^{(k)}(1) = 0, \quad A_{k-1}^{(k)}(1) = 1. \tag{6}$$

The validity of (5) follows from Pascal's formula

$$\binom{n}{p} = \binom{n-1}{p} + \binom{n-1}{p-1}$$

in its extended form for $n \geq 2$ and p any integer, with the understanding that $\binom{m}{q} = 0$ if $q > m \geq 1$, while $\binom{m}{q} = 0$ for $q < 0$ and $m \geq 1$.

In the next section we will prove Ramus' identity as a consequence of (5) and (6). Finally, in the last section of the paper, we will show how the recursions (5) lead to Konvalina and Liu's recursion (4).

2. Proof of Ramus' Identity

To obtain identity (2) from the system (5) and the conditions (6), we first attempt, in accordance with the theory of difference equations [3], to find solutions of the system in the form

$$A_j^{(k)}(n) = v_j \lambda^n, \quad j = 0, 1, \ldots, k-1, \tag{7}$$

for appropriate λ's and v_j's. Inserting (7) into (5), we find that

$$\begin{cases} v_j \lambda^n = v_j \lambda^{n-1} + v_{j+1} \lambda^{n-1}, \quad j = 0, 1, \ldots, k-2, \\ \text{and} \quad v_{k-1} \lambda^n = v_{k-1} \lambda^{n-1} + v_0 \lambda^{n-1}; \end{cases}$$

which will surely be satisfied if

$$\begin{cases} v_j \lambda = v_j + v_{j+1}, \quad j = 0, 1, \ldots, k-2, \\ \text{and} \quad v_{k-1} \lambda = v_{k-1} + v_0. \end{cases} \tag{8}$$

In matrix form the latter conditions assert that

$$\lambda \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \cdot \\ v_{k-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \cdot \\ v_{k-1} \end{pmatrix}. \tag{9}$$

In other words λ has to be an eigenvalue of the matrix on the right, with $v_0, v_1, v_2, \ldots, v_{k-1}$ being the components of an eigenvector corresponding to this eigenvalue.

Now the eigenvalues λ are the zeros of the $k \times k$ determinant

$$D(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1-\lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1-\lambda & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 0 & 0 & 0 & \cdots & 1-\lambda \end{vmatrix}. \tag{10}$$

Expanding, according to the first column, we find that

$$D(\lambda) = (1-\lambda)^k + (-1)^{k-1}. \tag{11}$$

Thus for λ to be a root of the determinant (10), $\lambda - 1$ will have to be a kth root of unity. Hence, the roots λ_p, $p = 0, 1, \ldots, k-1$, of (10) are given by

$$\lambda_p = 1 + \omega_p, \quad \text{where } \omega_p = e^{ip2\pi/k}, \quad p = 0, 1, \ldots, k-1. \tag{12}$$

Knowing what the eigenvalues of the matrix on the right of (9) are, one can determine the components v_j of the corresponding eigenvectors in accordance with the conditions (8). With $\lambda = \lambda_p = 1 + \omega_p$, these conditions read

$$\begin{cases} & (1+\omega_p)\, v_j = v_j + v_{j+1}, \quad j = 0, 1, \ldots, k-2, \\ \text{and} & (1+\omega_p)\, v_{k-1} = v_{k-1} + v_{0.} \end{cases}$$

The first $k-1$ conditions lead to $v_{j+1} = \omega_p v_j$, $j = 0, 1, \ldots, k-2$, so that $v_j = \omega_p^j v_0$, $j = 0, 1, \ldots, k-2$. The remaining condition leads to $v_{k-1} = \omega_p^{-1} v_0 = \omega_p^{k-1} v_0$. Thus, in all cases, the components v_j of the eigenvector corresponding to the eigenvalue $\lambda_p = 1 + \omega_p$ are given by

$$v_j = \omega_p^j v_{0,} \qquad j = 0, 1, \ldots, k-1, \tag{13}$$

where v_0 can be chosen arbitrarily.

If we now form the expressions

$$A_j(n) = v_j \lambda^n = c_p \omega_p^j (1+\omega_p)^n, \quad j = 0, 1, \ldots, k-1, \quad n \geq 1,$$

where the arbitrary constant c_p represents v_0 in (13), it then follows from the preceding calculations that these $A_j(n)$'s are solutions of the same system (5) that the $A_j^{(k)}(n)$'s satisfy. Superposing, the same is true of the expressions

$$A_j(n) = \sum_{p=0}^{k-1} c_p \omega_p^j (1+\omega_p)^n, \quad j = 0, 1, \ldots, k-1, \quad n \geq 1. \tag{14}$$

In view of the uniqueness of the solution of the system of recursion relations (5) satisfying the initial conditions (6), to prove that the $A_j^{(k)}(n)$'s are given by the formula (2), it will be sufficient to show that by choosing the c_p's in (14) all equal to $1/k$, the resulting expressions satisfy the initial conditions (6). In other words we need to show that

$$\frac{1}{k}\sum_{p=0}^{k-1} \omega_p^j (1+\omega_p) = \frac{1}{k}\sum_{p=0}^{k-1} \left(\omega_p^j + \omega_p^{j+1}\right) = \begin{cases} 0 \text{ if } j = 1, 2, \ldots, k-2 \\ 1 \text{ if } j = 0 \text{ or } j = k-1; \end{cases}$$

and this follows immediately by noting that

$$\sum_{p=0}^{k-1} \omega_p^l = \sum_{p=0}^{k-1} \omega_l^p = \left(1 - \omega_l^k\right)/(1-\omega_l) = 0 \quad \text{for } l = 1, 2, \ldots, k-1,$$

while as $\omega_p^l = 1$ if $l = 0$ or $l = k$,

$$\sum_{p=0}^{k-1} \omega_p^l = k \;\; \text{if } l = 0 \text{ or } l = k.$$

3. Konvalina and Liu's Recursion Relation

Here we want to indicate how the recursion relation (4) can be derived from our approach. To this end we introduce the operator E which sends the nth term $B(n)$ of a sequence into the $(n+1)^{st}$ term $B(n+1)$; the νth iterant of E, E^ν, then sends $B(n)$ into $B(n+\nu)$, which we write as

$$E^\nu B(n) = B(n+\nu).$$

Next, given a polynomial $P(x) = \sum_{\nu=0}^{q} a_\nu x^\nu$, we define the corresponding operator $P(E) = \sum_{\nu=0}^{q} a_\nu E^\nu$ in the obvious way as

$$P(E) B(n) = \sum_{\nu=0}^{q} a_\nu E^\nu B(n) = \sum_{\nu=0}^{q} a_\nu B(n+\nu).$$

Using this notation the recursion relations (5) can be written as

$$\begin{cases} \quad (1-E) A_j^{(k)}(n) + A_{j+1}^{(k)}(n) = 0 \quad j = 0, 1, \ldots, k-2, \\ \text{and} \quad (1-E) A_{k-1}^{(k)}(n) + A_0^{(k)}(n) = 0, \quad n \geq 1. \end{cases}$$

Viewing this as a $k \times k$ system of linear equations in the unknowns $A_j^{(k)}(n)$, $j = 0, 1, \ldots, k-1$, with operator coefficients, Cramer's rule, suitably interpreted [3, p. 112], is applicable; applying it, it follows that each $A_j^{(k)}(n)$, satisfies the recursion relation $D(E) A_j^{(k)}(n) = 0$, for $n \geq 1$, where

$$D(E) = \begin{vmatrix} 1-E & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1-E & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1-E & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 1-E \end{vmatrix}.$$

Clearly this determinant is the same as the determinant $D(\lambda)$ on the right of (10) with λ replaced by E. In (11) we found $D(\lambda) = (1-\lambda)^k + (-1)^{k-1}$. Hence, each $A_j^{(k)}(n)$ satisfies

$$\left[(1-E)^k + (-1)^{k-1}\right] A_j^{(k)}(n) = 0, \quad n \geq 1.$$

Expanding, this is equivalent to

$$A_j^{(k)}(n+k) = \sum_{l=1}^{k-1} (-1)^{l-1} \binom{k}{l} A_j^{(k)}(n+k-l) + \left[1 + (-1)^{k-1}\right] A_j^{(k)}(n), \quad n \geq 1,$$

precisely the recursion relation (4) of Konvalina and Liu.

References

[1] V. E. Hogatt and G. L. Alexanderson, *Sums of Partition Sets in Generalized Pascal Triangles I*, The Fibonacci Quarterly, 14.2 (1976), 117–125.

[2] J. Konvalina and Y.-H. Liu, *Arithmetic Progression Sums of Binomial Coefficients*, Appl. Math Lett., 10 (1997), 11–13.

[3] H. Levy and F. Lessman, *Finite Difference Equations*, The Macmillan Company, New York, 1961.

[4] C. Ramus, *Solution Générale d'un Problème d'Analyse Combinatoire*, J. Reine Angew. Math., 11 (1834), 353–355.

[5] E. Netto, *Lehrbuch der Combinatorik*, 2nd ed., Teubner, Berlin, 1927.

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