ON HOFSTADTER'S MARRIED FUNCTIONS

THOMAS STOLL

Abstract. In this note we show that Hofstadter's married functions generated by the intertwined system of recurrences $a(0) = 1$, $b(0) = 0$, $b(n) = n - a(b(n-1))$, $a(n) = n - b(a(n-1))$ has the solutions $a(n) = \lfloor (n+1)\phi^{-1} \rfloor + \varepsilon_1(n)$ and $b(n) = \lfloor (n+1)\phi^{-1} \rfloor - \varepsilon_2(n)$, where ϕ is the golden ratio and $\varepsilon_1, \varepsilon_2$ are indicator functions of Fibonacci numbers diminished by 1.

1. Introduction

In his well–known book "Gödel, Escher, Bach: An Eternal Golden Braid," D. R. Hofstadter [9] introduces several recurrences which give rise to particularly intriguing integer sequences. Mention, for instance, the famous *Hofstadter's Q–sequence* (also known as *Meta-Fibonacci sequence* [6], entry A005185 in Sloane's Encyclopedia [13]) which is defined by $Q(1) = Q(2) = 1$ and

$$Q(n) = Q(n - Q(n-1)) + Q(n - Q(n-2)) \quad \text{for } n > 2. \tag{1.1}$$

Each term of the sequence is the sum of two preceding terms but, in contrast to the Fibonacci sequence, not necessarily the two last terms. The sequence $Q(n)$ shows an erratic behavior as well as a certain degree of regularity (see [12]). Another link to the sequence of Fibonacci numbers $(F_k)_{k\geq 1} = 1, 1, 2, 3, 5, 8, 13\ldots$ is given in *Hofstadter's G–sequence* (A005206), which is generated by $G(0) = 0$ and

$$G(n) = n - G(G(n-1)) \quad \text{for } n > 0. \tag{1.2}$$

It has been shown [11] that if $n = F_{r(1)} + F_{r(2)} + \cdots + F_{r(m)}$ is the Zeckendorf expansion of n, then $G(n) = F_{r(1)-1} + F_{r(2)-1} + \cdots + F_{r(m)-1}$. Furthermore, Downey/Griswold [1] and Granville/Rasson [3] proved an explicit formula for $G(n)$, namely,

$$G(n) = \lfloor (n+1)\mu \rfloor, \tag{1.3}$$

where $\mu = (\sqrt{5}-1)/2 = \phi^{-1}$ and $\phi = (\sqrt{5}+1)/2$ is the golden ratio. Various generalizations of (1.2) have been investigated in [1, 10].

In this note we are concerned with *Hofstadter's "married functions"* (or *Hofstadter's male–female sequences* [14]) defined by two intertwined functions $a(n)$ and $b(n)$ (see p.137 of [9]).

Definition 1.1. *Denote by $a(n)$ and $b(n)$ the sequences defined by $a(0) = 1$, $b(0) = 0$ and for $n > 0$ by the intertwined system of recurrences*

$$\begin{cases} b(n) = n - a(b(n-1)), \\ a(n) = n - b(a(n-1)). \end{cases} \tag{1.4}$$

The first one hundred terms of the "female" sequence $a(n)$ (A005378) and, respectively, of the "male" sequence $b(n)$ (A005379) are given in Table 1.

Table 1. Values of $a(n)$, $b(n)$ for $0 \le n \le 99$

n	$a(n)$	$b(n)$	n	$a(n)$	$b(n)$	n	$a(n)$	$b(n)$	n	$a(n)$	$b(n)$
0	1	0	25	16	16	50	31	31	75	46	46
1	1	0	26	16	16	51	32	32	76	47	47
2	2	1	27	17	17	52	32	32	77	48	48
3	2	2	28	17	17	53	33	33	78	48	48
4	3	2	29	18	18	54	34	33	79	49	49
5	3	3	30	19	19	55	34	34	80	50	50
6	4	4	31	19	19	56	35	35	81	50	50
7	5	4	32	20	20	57	35	35	82	51	51
8	5	5	33	21	20	58	36	36	83	51	51
9	6	6	34	21	21	59	37	37	84	52	52
10	6	6	35	22	22	60	37	37	85	53	53
11	7	7	36	22	22	61	38	38	86	53	53
12	8	7	37	23	23	62	38	38	87	54	54
13	8	8	38	24	24	63	39	39	88	55	54
14	9	9	39	24	24	64	40	40	89	55	55
15	9	9	40	25	25	65	40	40	90	56	56
16	10	10	41	25	25	66	41	41	91	56	56
17	11	11	42	26	26	67	42	42	92	57	57
18	11	11	43	27	27	68	42	42	93	58	58
19	12	12	44	27	27	69	43	43	94	58	58
20	13	12	45	28	28	70	43	43	95	59	59
21	13	13	46	29	29	71	44	44	96	59	59
22	14	14	47	29	29	72	45	45	97	60	60
23	14	14	48	30	30	73	45	45	98	61	61
24	15	15	49	30	30	74	46	46	99	61	61

A simple inductive argument shows that $0 < a(n) \le n+1$ and $0 \le b(n) \le n$, thus ensuring that both sequences are well-defined for all $n \ge 0$ by the recursion (1.4). J. Grytczuk [4, 5] provided a general framework for the recursions (1.2) and (1.4), namely by linking them to Richelieu cryptosystems and symbolic dynamics on infinite words. We here prove explicit formulas for $a(n)$ and $b(n)$ which, somehow as a surprise, involve both the quantity μ from formula (1.3) and the explicit notion of Fibonacci numbers F_k.

Our main result is the following theorem.

Theorem 1.2. *For all $n \ge 0$ there hold*

$$a(n) = \lfloor (n+1)\mu \rfloor + \varepsilon_1(n),$$
$$b(n) = \lfloor (n+1)\mu \rfloor - \varepsilon_2(n),$$

where for $k \ge 1$,

$$\varepsilon_1(n) = \begin{cases} 1, & \text{if } n = F_{2k} - 1; \\ 0, & \text{else.} \end{cases} \qquad \varepsilon_2(n) = \begin{cases} 1, & \text{if } n = F_{2k+1} - 1; \\ 0, & \text{else.} \end{cases}$$

2. Proof

The proof of Theorem 1.2 involves an inductive argument, where several cases have to be taken into account. We begin with two lemmas which link the floor expression of the statement (a so-called *Beatty sequence*) with the Fibonacci numbers.

Lemma 2.1. *Let $n \geq 4$. Then*

(1) $\lfloor n\mu \rfloor = F_{2k} - 1 \Leftrightarrow n = F_{2k+1} - 1$.
(2) $\lfloor n\mu \rfloor = F_{2k-1} - 1 \Leftrightarrow n = F_{2k} - 1$ *or* $n = F_{2k}$.
(3) $\lfloor n\mu \rfloor = F_{2k} \Leftrightarrow n = F_{2k+1}$ *or* $n = F_{2k+1} + 1$.
(4) $\lfloor n\mu \rfloor = F_{2k-1} \Leftrightarrow n = F_{2k} + 1$.

Proof. The golden ratio $\phi = 1.6180339\ldots$ has the simple continued fraction expansion $[1, 1, 1, \ldots]$. Thus, for any $k \geq 2$, the quotient

$$\frac{F_k}{F_{k-1}} = 1 + \frac{1}{\frac{F_{k-1}}{F_{k-2}}} = \cdots = \left[1, 1, \ldots, \frac{F_2}{F_1}\right] = [1, 1, 1, \ldots, 1]$$

is just the $(k-1)$th convergent of ϕ. Consequently, general theory of continued fractions (e.g., see Theorem 163 and Theorem 171 in [8]) yields

$$\phi - \frac{F_k}{F_{k-1}} = \frac{(-1)^k}{F_{k-1}(\phi F_{k-1} + F_{k-2})}.$$

Since $\phi F_{k-1} + F_{k-2} \leq (\phi + 1)F_{k-1} < 3F_{k-1}$ and $\phi F_{k-1} + F_{k-2} > F_{k-1}$, we have the estimates

$$\frac{1}{3F_{2k-1}^2} < \phi - \frac{F_{2k}}{F_{2k-1}} < \frac{1}{F_{2k-1}^2} \quad \text{and} \tag{2.1}$$

$$\frac{1}{3F_{2k}^2} < \frac{F_{2k+1}}{F_{2k}} - \phi < \frac{1}{F_{2k}^2}. \tag{2.2}$$

First, consider case (1). Obviously, the left hand equation is equivalent to $F_{2k} - 1 \leq n\phi^{-1} < F_{2k}$ for $k \geq 2$, which again can be rewritten as

$$0 < \phi F_{2k} - n \leq \phi. \tag{2.3}$$

We have to prove that this inequality holds if and only if $n = F_{2k+1} - 1$. Instead of directly proving (1), we show the inequality

$$\phi - 1 < \phi F_{2k} - (F_{2k+1} - 1) \leq 1, \tag{2.4}$$

which implies (1). Namely, since $0 < \phi - 1$ and $1 < \phi$ and (2.3), this shows the "$\Leftarrow$" part of (1). On the other hand, the bounds in (2.3) ensure that $\phi F_{2k} - F_{2k+1} \leq 0$ and $\phi F_{2k} - (F_{2k+1} - 2) > \phi$, so that by linearity there are no other values of n satisfying (2.3). This give the "$\Rightarrow$" part of (1). Now, as for the proof of (2.4), we see that by (2.2),

$$\phi - 1 < 1 - \frac{1}{F_{2k}} < 1 + F_{2k}\left(\phi - \frac{F_{2k+1}}{F_{2k}}\right) < 1 - \frac{1}{3F_{2k}} < 1,$$

where the first and last inequality holds by trivial means and the middle term corresponds to the middle term in (2.4). This finishes the proof of (1).

Similarly, in case (2) we have

$$0 < \phi F_{2k-1} - n \leq \phi \tag{2.5}$$

for $k \geq 3$ and it is sufficient to show that $0 < \phi F_{2k-1} - F_{2k} \leq \phi - 1$, since then only $n = F_{2k} - 1$ and $n = F_{2k}$ can satisfy (2.5). Here, (2.1) yields

$$0 < \frac{1}{3F_{2k-1}} < F_{2k-1}\left(\phi - \frac{F_{2k}}{F_{2k-1}}\right) < \frac{1}{F_{2k-1}} < \phi - 1,$$

which gives the equivalence in case (2).

For case (3) it suffices to ensure that $0 \leq F_{2k+1} - \phi F_{2k} < \phi - 1$ for $k \geq 2$, which holds true due to

$$0 < \frac{1}{3F_{2k}} < F_{2k}\left(\frac{F_{2k+1}}{F_{2k}} - \phi\right) < \frac{1}{F_{2k}} < \phi - 1.$$

Finally, in case (4) the condition $\phi - 1 \leq F_{2k} + 1 - \phi F_{2k-1} < 1$ is guaranteed for $k \geq 3$ by

$$\phi - 1 < 1 - \frac{1}{F_{2k-1}} < 1 + F_{2k-1}\left(\frac{F_{2k}}{F_{2k-1}} - \phi\right) < 1 - \frac{1}{3F_{2k-1}} < 1$$

and by directly checking (4) for $k = 2$. □

In order to get our inductive argument to work later, we apply Lemma 2.1 to evaluate the indicator functions ε_1, ε_2 at $\lfloor n\mu \rfloor$. Obviously, by definition,

$$\varepsilon_1(0) = \varepsilon_2(1) = 1, \qquad \varepsilon_1(1) = \varepsilon_2(0) = 0. \tag{2.6}$$

Moreover, it is clear from Lemma 2.1, cases (1) and (2), that for all $n \geq 0$ there hold

$$\varepsilon_1(\lfloor n\mu \rfloor) = \begin{cases} 1, & \text{if } n = F_{2k+1} - 1; \\ 0, & \text{else.} \end{cases} \tag{2.7}$$

$$\varepsilon_2(\lfloor n\mu \rfloor) = \begin{cases} 1, & \text{if } n = F_{2k} - 1 \text{ or } n = F_{2k}; \\ 0, & \text{else.} \end{cases} \tag{2.8}$$

Our second lemma concerns nested Beatty sequences of type $\lfloor \mu \lfloor \mu n \rfloor + C \rfloor$ and will be useful to get rid of subsequent floors at specified points in the induction step.

Lemma 2.2.

(1) *For all $n \geq 0$ holds $\lfloor \mu n + \mu \rfloor = n - \lfloor \mu \lfloor \mu n \rfloor + \mu \rfloor$.*
(2) *For $n = F_{2k}$ with $k \geq 2$ holds $\lfloor \mu n + \mu \rfloor = n - \lfloor \mu \lfloor \mu n \rfloor + 2\mu \rfloor$.*
(3) *For $n = F_{2k+1}$ with $k \geq 2$ holds $\lfloor \mu n + \mu \rfloor = n - \lfloor \mu \lfloor \mu n \rfloor \rfloor - 1$.*

Proof. Equation (1) is Lemma 1 in [1]. We proceed with the proof of (2) and (3). First, let $k \geq 3$. Concerning statement (2), we observe with help of Lemma 2.1 (cases (2), (3), (4) therein), that

$$\begin{aligned} F_{2k} - \lfloor \mu \lfloor \mu F_{2k} \rfloor + 2\mu \rfloor &= F_{2k} - \lfloor \mu (F_{2k-1} + 1) \rfloor \\ &= F_{2k} - F_{2k-2} = F_{2k-1} = \lfloor \mu (F_{2k} + 1) \rfloor. \end{aligned}$$

Similarly, for statement (3) we use Lemma 2.1 (cases (3), (2), (3), respectively) and obtain

$$\begin{aligned} F_{2k+1} - \lfloor \mu \lfloor \mu F_{2k+1} \rfloor \rfloor - 1 &= F_{2k+1} - \lfloor \mu F_{2k} \rfloor - 1 \\ &= F_{2k+1} - (F_{2k-1} - 1) - 1 \\ &= F_{2k} = \lfloor \mu (F_{2k+1} + 1) \rfloor. \end{aligned}$$

The case $k = 2$ is easily checked by hand. □

We are now ready for the proof of Theorem 1.2.

Proof of Theorem 1.2. To begin with, we observe that the formulas given in Theorem 1.2 are true if $n \in \{0,1,2\}$ (e.g., compare with Table 1). Suppose now $n \geq 3$. According to (1.4) we have to show that

$$\begin{aligned}\lfloor (n+1)\mu \rfloor - \varepsilon_2(n) = n - {} & \lfloor \mu\left(\lfloor n\mu \rfloor - \varepsilon_2(n-1) + 1\right)\rfloor \\ & - \varepsilon_1\left(\lfloor n\mu \rfloor - \varepsilon_2(n-1)\right),\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\lfloor (n+1)\mu \rfloor + \varepsilon_1(n) = n - {} & \lfloor \mu\left(\lfloor n\mu \rfloor + \varepsilon_1(n-1) + 1\right)\rfloor \\ & + \varepsilon_2\left(\lfloor n\mu \rfloor + \varepsilon_1(n-1)\right).\end{aligned} \tag{2.10}$$

We distinguish several cases on n.

First, if $n \neq F_{2k}, F_{2k}-1, F_{2k+1}, F_{2k+1}-1$ then by definition of the indicator functions,

$$\varepsilon_1(n) = \varepsilon_1(n-1) = \varepsilon_2(n) = \varepsilon_2(n-1) = 0.$$

Furthermore, by (2.7) and (2.8),

$$\varepsilon_1(\lfloor n\mu \rfloor) = \varepsilon_2(\lfloor n\mu \rfloor) = 0.$$

Then, the equalities (2.9) and (2.10) hold by (1) of Lemma 2.2.

If $n = F_{2k+1} - 1$, then

$$\begin{aligned}&\varepsilon_1(n) = \varepsilon_1(n-1) = \varepsilon_2(n-1) = \varepsilon_2(\lfloor n\mu \rfloor) = 0 \quad \text{and}\\ &\varepsilon_2(n) = \varepsilon_1(\lfloor n\mu \rfloor) = 1\end{aligned}$$

and again (1) from Lemma 2.2 yields (2.9) and (2.10). In the same style, for $n = F_{2k} - 1$ we have

$$\begin{aligned}&\varepsilon_1(n) = \varepsilon_2(\lfloor n\mu \rfloor) = 1 \quad \text{and}\\ &\varepsilon_1(n-1) = \varepsilon_2(n) = \varepsilon_2(n-1) = \varepsilon_1(\lfloor n\mu \rfloor) = 0\end{aligned}$$

and (2.9), (2.10) by (1) from Lemma 2.2. Moreover, for $n = F_{2k}$ we have

$$\varepsilon_2(n) = \varepsilon_2(n-1) = \varepsilon_1(\lfloor n\mu \rfloor) = 0$$

and (2.9) again by (1). On the other hand, for $n = F_{2k+1}$ it follows

$$\varepsilon_1(n) = \varepsilon_1(n-1) = \varepsilon_2(\lfloor n\mu \rfloor) = 0$$

and (2.10).

It remains to prove (2.9) for $n = F_{2k+1}$ and (2.10) for $n = F_{2k}$. For suppose $n = F_{2k+1}$, then $\varepsilon_2(n) = 0$ and

$$\varepsilon_2(n-1) = \varepsilon_1(\lfloor n\mu \rfloor - 1) = 1.$$

Now, statement (3) of Lemma 2.2 gives equation (2.9). Finally, if $n = F_{2k}$ then $\varepsilon_1(n) = \varepsilon_2(\lfloor n\mu \rfloor + 1) = 0$ and $\varepsilon_1(n-1) = 1$, and (2.10) corresponds to (2) of Lemma 2.2. This finishes the proof of Theorem 1.2. □

3. Epilogue

It would be interesting to know whether there exist explicit formulas for (what we may call) *Hofstadter's parents-child sequences* $a(n)$, $b(n)$ and $c(n)$ defined by $a(0) = 1$, $b(0) = c(0) = 0$ and

$$\begin{cases} b(n) = n - c(b(n-1)), \\ a(n) = n - b(a(n-1)), \\ c(n) = n - a(c(n-1)) \qquad \text{for } n > 0. \end{cases} \tag{3.1}$$

Consider, for instance, the "mother" sequence

$$a(n) = 1, 0, 2, 1, 3, 3, 4, 4, 5, 7, 7, 8, 7, 10, 8, 10, \ldots.$$

We are surprised by the fact that $a(n) = \lfloor (n+1)\mu \rfloor$ for $4 \le n \le 8$, $30 \le n \le 40$, $140 \le n \le 176$, $606 \le n \le 752$ etc., while on all other intervals irregular oscillations can be observed.

Acknowledgment

The author is recipient of an APART-fellowship of the Austrian Academy of Sciences at the University of Waterloo, Canada. Support has also been granted by the Austrian Science Foundation FWF, Grant S9604.

References

[1] P. J. Downey and R. E. Griswold,On a Family of Nested Recurrences, Fibonacci Quart. 22.4 (1984), 310–317.
[2] P. Erdős and R. L. Graham, *Old and New Problems and Results in Combinatorial Number Theory*, L'Enseignement Mathématique, Univ. Genève, Geneva, 1980.
[3] V. Granville and J.-P. Rasson, *A Strange Recursive Relation*, J. Number Theory, 30.2 (1988), 238–241.
[4] J. Grytczuk, *Intertwined Infinite Binary Words*, Discrete Appl. Math., 66.1 (1996), 95–99.
[5] J. Grytczuk, *Infinite Self-similar Words*,Discrete Math., 161.1-3 (1996), 133–141.
[6] R. K. Guy, *Some Suspiciously Simple Sequences*, Amer. Math. Monthly, 93 (1986), 186–191.
[7] R. K. Guy, *Three Sequences of Hofstadter, Chapter E31*, Unsolved Problems in Number Theory, Third edition, Springer, New York, 2004.
[8] G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, Fifth edition, Clarendon Press, Oxford, 1979.
[9] D. R. Hofstadter, *Gödel, Escher, Bach: An Eternal Golden Braid*, Basic Books, New York, 1979.
[10] P. Kiss and B. Zay, *On a Generalization of a Recursive Sequence*, Fibonacci Quart. 30.2 (1992), 103–109.
[11] D. S. Meek and G. H. J. Van Rees, *The Solution of an Iterated Recurrence*, Fibonacci Quart. 22.2 (1984), 101–104.
[12] K. Pinn,*Order and Chaos in Hofstadter's $Q(n)$ Sequence*, Complexity 4.3 (1999), 41–46.
[13] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at `http://www.research.att.com/~njas/sequences`, 1996–2006.
[14] E. W. Weisstein, Hofstadter Male-Female Sequence (from Mathworld), published electronically at `http://mathworld.wolfram.com/HofstadterMale-FemaleSequences.html`, 1999–2006.

MSC2000: 11B37, 11B39.

Institute of Discrete Mathematics and Geometry, TU Vienna, Wiedner Hauptstrasse 8–10, 1040 Vienna, Austria
E-mail address: stoll@dmg.tuwien.ac.at