DISJOINT COVERING OF $\mathbb{N}$ BY A HOMOGENEOUS LINEAR RECURRENCE

NECULAE VORNICESCU

Abstract. We prove that a homogeneous linear recurrence with positive coefficients can generate a disjoint covering of $\mathbb{N}$.

1. Introduction

In [2], a linear recurrence is said to *generate a disjoint covering* of $\mathbb{N} = \{1, 2, \ldots\}$ if there exists a family of recurring sequences such that each $n \in \mathbb{N}$ occurs in exactly one sequence of this family. In [3], Simpson proved that an arithmetic progression with positive terms can generate a disjoint covering of $\mathbb{N}$. In [1], Ando and Hilano proved that a linear recurrence, whose characteristic equation has a Pisot number root, can generate a disjoint covering of $\mathbb{N}$. In [2], Burke and Bergum proved that a linear recurrence, whose characteristic equation has a prime root, can generate a disjoint covering of $\mathbb{N}$. In [4], Zöllner proved that the Fibonacci recurrence can generate a disjoint covering of $\mathbb{N}$.

The result of this article is that a homogeneous linear recurrence with positive coefficients can generate a disjoint covering of $\mathbb{N}$.

2. Disjoint Covering of $\mathbb{N}$ with Sequences Verifying a Homogeneous Linear Recurrence

Consider the homogeneous linear recurrence of order $m, m \geq 2$,

$$x_n = a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_m x_{n-m}, \tag{1}$$

where $a_1, a_m \in \mathbb{N}$ and $a_i \in \mathbb{N} \cup \{0\}$ for $1 < i \leq m-1$. In the case $m = 1$ we have a geometric progression and the result is trivial.

Given $y_1, \ldots, y_m \in \mathbb{R}$, we denote by $S(y_1, y_2, \ldots, y_m)$ the sequence $\{x_n\}$ such that

$$\begin{aligned} x_n &= y_n, \quad \text{for} \quad n = 1, 2, \ldots, m, \\ x_n &= a_1 x_{n-1} + a_2 x_{n-2} + \cdots + a_m x_{n-m}, \quad \text{for} \quad n > m. \end{aligned}$$

The elements $x_1, x_2, \ldots, x_m$ are called *initial elements* of the sequence $S(x_1, x_2, \ldots, x_m)$. In the sequel we will consider only sequences with elements in $\mathbb{N}$.

In the following two lemmas we give some properties of the sequences defined above.

Lemma 1. *If $x_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$, such that $x_1 < x_2 < \cdots < x_m$, then $S(x_1, x_2, \ldots, x_m)$ is an increasing sequence and if x_{s+1}, x_{t+1} with $s < t$ are not initial elements, then*

$$x_{t+1} - x_{s+1} > x_t - x_s.$$

Proof. For $n > m$, $x_n > a_1 x_{n-1} \geq x_{n-1}$. For $m < s < t$, $x_{t+1} - x_{s+1} > a_1(x_t - x_s) \geq x_t - x_s$ □

Lemma 2. *Let $S_1 = \{x_n\} = S(x_1, x_2, \ldots, x_m)$ and $S_2 = \{y_n\} = S(y_1, y_2, \ldots, y_m)$ be two sequences satisfying*

1. $x_1 < x_2 < \cdots < x_m$
2. *There exists $j \geq 1$ such that $x_{j+n-1} < y_n < x_{j+n}$, $n = 1, 2, \ldots, m$.*

Then the following statements are true.

(a) *For all $n \in \mathbb{N}$ we have*

$$x_{j+n-1} < y_n < x_{j+n}. \tag{2}$$

(b) *If x_{s+1} and y_{t+1} are not initial elements (that is, $s, t \geq m$) then*

$$\begin{aligned} & x_s < y_t \quad \textit{implies} \quad y_{t+1} - x_{s+1} > y_t - x_s \\ \textit{and} \quad & y_t < x_s \quad \textit{implies} \quad x_{s+1} - y_{t+1} > x_s - y_t \end{aligned}$$

Proof.

(a) For $n > m$ inequalities (2) can be easily established by an induction argument, using the recurrence relation (1).

(b) For $n = t$, inequalities (2) becomes

$$x_{j+t-1} < y_t < x_{j+t}. \tag{3}$$

If $x_s < y_t$, then from (3) we deduce that $x_s \leq x_{j+t-1}$. From Lemma 1 and from (2) we obtain

$$x_{s-i} \leq x_{j+t-1-i} < y_{t-i},$$

for each $i = 0, 1, \ldots, m-1$. Then

$$\begin{aligned} y_{t+1} - x_{s+1} &= a_1(y_t - x_s) + a_2(y_{t-1} - x_{s-1}) + \cdots + a_m(y_{t+1-m} - x_{s+1-m}) \\ &> a_1(y_t - x_s) \geq y_t - x_s. \end{aligned}$$

If $y_t < x_s$, then from (3) we deduce that $x_s \geq x_{j+t}$ and further $x_{s-i} \geq x_{j-i+t} > y_{t-i}$ for each $i = 0, 1, \ldots, m-1$. Thus, $x_{s+1} - y_{t+1} > a_1(x_s - y_t) \geq x_s - y_t$.

The main result is the next theorem. □

Theorem 3. *If $m \geq 2$, $a_1, a_m \in \mathbb{N}$ and $a_i \in \mathbb{N} \cup \{0\}$ for $1 < i \leq m-1$, then the recurrence relation (1) can generate a family of sequences $\{S_k\}_{k \in \mathbb{N}}$, which is a disjoint covering of $\mathbb{N}$.*

Proof. We will construct the family $\{S_k\}$ by induction.

$$\begin{aligned} S_1 &= \{x_n^1\} = S(1, 2, \ldots, m), \\ S_2 &= \{x_n^2\} = S(x_1^2, x_2^2, \ldots, x_m^2), \end{aligned}$$

where $x_1^2, x_2^2, \ldots, x_m^2$ are defined as follows. The number x_1^2 is the smallest natural number not in S_1. Hence, there exists i such that $x_i^1 = x_1^2 - 1 \in S_1$. Then $x_2^2 = x_{i+1}^1 + 1, x_3^2 = x_{i+2}^1 + 1, \ldots, x_m^2 = x_{i+m-1}^1 + 1$.

From the choice of x_1^2 we deduce that $1, 2, \ldots, x_i^1 \in S_1$, $x_i^1 + 1 \notin S_1$, and $i \geq m$.

We will prove that

$$x_{i+n-1}^1 < x_n^2 < x_{i+n}^1, \quad \text{for} \quad n = 1, 2, \ldots, m. \tag{4}$$

Since $x_n^2 = x_{i+n-1}^1 + 1$, in order to prove (4), it is enough to show that

$$x_{i+n}^1 > x_{i+n-1}^1 + 1, \quad \text{for} \quad n = 1, 2, \ldots, m. \tag{5}$$

We prove (5) by induction. For $n = 1$, since $x_i^1 + 1 \notin S_1$ we have $x_{i+1}^1 > x_i^1 + 1$. We suppose that, for some r, $1 \leq r < m$, we have $x_{i+r}^1 > x_{i+r-1}^1 + 1$. Since $i \geq m$, x_{i+r}^1 is not an initial element of S_1 and applying Lemma 1 we conclude that

$$x_{i+r+1}^1 - x_{i+r}^1 > x_{i+r}^1 - x_{i+r-1}^1 > 1.$$

Hence, (4) and (5) hold.

From Lemma 2 it follows that

$$x_{i+n-1}^1 < x_n^2 < x_{i+n}^1, \quad \text{for all} \quad n \in \mathbb{N},$$

and, in the case when x_{s+1}^1, x_{t+1}^2 are not initial elements, we have

$$\begin{array}{lll} x_s^1 < x_t^2 & \text{implies} & x_{t+1}^2 - x_{s+1}^1 > x_t^2 - x_s^1 \qquad \text{and} \\ x_s^1 > x_t^2 & \text{implies} & x_{s+1}^1 - x_{t+1}^2 > x_s^1 - x_t^2. \end{array}$$

Now we suppose that, for some $k \geq 2$, we have constructed a family of sequences $S_j = \{x_n^j\} = S(x_1^j, x_2^j, \ldots, x_m^j)$, $j = 1, 2, \ldots, k$, satisfying the following properties.

P1) For each j, $2 \leq j \leq k$, x_1^j is the smallest natural number not yet covered by the sequences $S_1, S_2, \ldots, S_{j-1}$ and if $x_s^i = x_1^j - 1$, with some $i = 1, 2, \ldots, j-1$ and $s \in \mathbb{N}$, then

$$x_2^j = x_{s+1}^1 + 1, x_3^j = x_{s+2}^1 + 1, \ldots, x_m^j = x_{s+m-1}^1 + 1.$$

P2) For any j_1, j_2, $1 \leq j_1 < j_2 \leq k$, the sequences S_{j_1} and S_{j_2} are disjoint and there exists $r \geq 1$ such that

$$x_r^{j_1} < x_1^{j_2} < x_{r+1}^{j_1} < x_2^{j_2} < \cdots < x_{r+n-1}^{j_1} < x_n^{j_2} < x_{r+n}^{j_1} < \cdots .$$

This means that, for any two sequences S_{j_1} and S_{j_2}, with S_{j_2} having the greatest first element, all other elements of S_{j_2} are individually separated by individual elements of S_{j_1}.

For constructing the sequence S_{k+1} we need to consider the set $Z_k \subset \mathbb{N}$ as the set covered by the sequences $S_j, j = 1, 2, \ldots, k$ and the function $F\colon Z_k \to Z_k$ defined as $F(x_s^j) = x_{s+1}^j$, for $x_s^j \in Z_k$. We denote by F_n the composed function $F_n = F(F_{n-1})$, $n > 1$, where $F_1 = F$ and $F_0(x) = x$ for $x \in Z_k$. We say that $x \in Z_k$ is an initial element if x is an initial element of the sequence S_j which contains x.

Next we show two properties of the set Z_k and of the function F.

P3) If $x \in Z_k$, $x > 1$ and if $x - 1 \notin Z_k$, then x is not an initial element. Indeed, if we suppose that x is an initial element then there exists j, $1 \leq j \leq k$ such that $x \in \{x_1^j, x_2^j, \ldots, x_m^j\}$. If $j > 1$, then from P1 we deduce that $x_1^j - 1, x_2^j - 1, \ldots, x_m^j - 1 \in Z_k$, which contradicts the hypothesis. If $j = 1$, then $x \in \{2, 3, \ldots, m\}$ and again $x - 1 \in Z_k$.

P4) If $x, y \in Z_k$ with $x < y$, then $F(x) < F(y)$ and if, in addition, x and y are not initial elements, then

$$F(y) - F(x) > y - x.$$

Indeed, if there exists j, $1 \leq j \leq k$ such that $x, y \in S_j$, then Property P4) results from Lemma 1. If $x \in S_{j_1}$ and $y \in S_{j_2}$ with $j_1 \neq j_2$, then hypotheses 1 and 2 of Lemma 2 are satisfied, hence, P4) results from the conclusions of this lemma.

We now construct the sequence $S_{k+1} = \{x_n^{k+1}\} = S(x_1^{k+1}, x_2^{k+1}, \ldots, x_m^{k+1})$ as follows:

$$\begin{aligned} &x_1^{k+1} \quad \text{is the smallest natural number not in } Z_k \\ &x_2^{k+1} = F(x_1^{k+1} - 1) + 1, \\ &x_3^{k+1} = F_2(x_1^{k+1} - 1) + 1, \\ &\ldots \\ &x_m^{k+1} = F_{m-1}(x_1^{k+1} - 1) + 1. \end{aligned}$$

We will show that the family of sequences $S_j, j = 1, 2, \ldots, k+1$, satisfies Properties P1) and P2).

Obviously, x_1^{k+1} is the smallest natural number not covered by the sequences $S_1, S_2, \ldots, S_k$.

Let $x_s^i = x_1^{k+1} - 1 \in Z_k$ with $1 \leq i \leq k$ and $s \in \mathbb{N}$. Then, from the definition of F and F_n we deduce that

$$x_2^{k+1} = x_{s+1}^i + 1, x_3^{k+1} = x_{s+2}^i + 1, \ldots, x_m^{k+1} = x_{s+m-1}^i + 1,$$

and therefore P1) is verified.

Let us denote by E the set $E = \{x_s^i - k + 1, x_s^i - k + 2, \ldots, x_s^i\}$. From the choice of x_1^{k+1} it follows that $E \subset Z_k$ and the first element of each sequence $S_j, j = 1, 2, \ldots, k$ is less than x_1^{k+1}.

We claim that each sequence $S_j, j = 1, 2, \ldots, k$, contains exactly one element from E. Indeed, if we suppose that there exists a sequence S_{j_1}, for some $j_1, 1 \leq j_1 \leq k$, containing two elements such that

$$x_s^i - k + 1 \leq x_r^{j_1} < x_{r+1}^{j_1} \leq x_s^i,$$

then there exists a sequence S_{j_2}, for some $j_2, 1 \leq j_2 \leq k$ such that $S_{j_2} \cap E = \emptyset$. Let $x_t^{j_2}$ be the largest elements of S_{j_2} such that $x_t^{j_2} < x_s^i - k + 1$. Then $x_{t+1}^{j_2} > x_s^i$ and $x_t^{j_2} < x_r^{j_1} < x_{r+1}^{j_1} < x_{t+1}^{j_2}$, a fact which contradicts P2).

Further, from P2) we obtain

$$\begin{aligned}
&x_s^i - k + 1 < x_s^i - k + 2 < \cdots < x_s^i \\
&< F(x_s^i - k + 1) < F(x_s^i - k + 2) < \cdots < F(x_s^i) \\
&< F_2(x_s^i - k + 1) < \cdots < F_2(x_s^i) < F_3(x_s^i - k + 1) < \cdots \\
&< F_n(x_s^i - k + 1) < \cdots < F_n(x_s^i) < F_{n+1}(x_s^i - k + 1) < \cdots . \qquad (6)
\end{aligned}$$

Hence,

$$Z_k = \{1, 2, \ldots, x_s^i\} \cup \left(\bigcup_{n=1}^{\infty} F_n(E) \right).$$

Let us remark that all natural numbers less than x_1^{k+1} belong to Z_k.

Now we show that

$$F_p(x_s^i) < x_{p+1}^{k+1} < F_{p+1}(x_s^i - k + 1), \text{ for all } p = 0, 1, \ldots, m-1, \qquad (7)$$

where $F_0(x_s^i) = x_s^i$. Since $x_{p+1}^{k+1} = x_{s+p}^i + 1 = F_p(x_s^i) + 1$, in order to prove (7), it is enough to show that

$$F_{p+1}(x_s^i - k + 1) - F_p(x_s^i) > 1, \text{ for all } p = 0, 1, \ldots, m-1. \qquad (8)$$

From $x_1^{k+1} = x_s^i + 1 \notin Z_k$, and since each sequence S_j, $j = 1, 2, \ldots, k$ contains exactly one element from E, we deduce that $F(x_s^i - k + 1) > x_1^{k+1}$ and $F(x_s^i - k + 1) - 1 \notin Z_k$. Hence, (8) holds for $p = 0$ and $F(x_s^i - k + 1)$ is not an initial element in Z_k. We will proceed by induction. We suppose that the inequality (8) holds for some $p = 0, 1, \ldots, m-2$. Then from (6) we obtain $F_{p+1}(x_s^i - k + 1)$ is not an initial element. We need to consider two cases.

A. $F_p(x_s^i)$ is not an initial element. From P4) it follows that

$$F_{p+2}(x_s^i - k + 1) - F_{p+1}(x_s^i) > F_{p+1}(x_s^i - k + 1) - F_p(x_s^i) > 1.$$

Hence, the inequalities in (7) hold for $p + 1$ and therefore, by induction, they hold for all $0 \leq p \leq m - 1$.

B. $F_p(x_s^i)$ is an initial element in Z_k. Let q, $0 \leq q \leq k - 2$, be such that $F_p(x_s^i - t)$ is an initial element for $t = 0, \ldots, q$ and $F_p(x_s^i - q - 1)$ is not an initial element. Then from P1) we get

$$F_{p+1}(x_s^i) = F_{p+1}(x_s^i - 1) + 1 = F_{p+1}(x_s^i - 2) + 2 = \cdots = F_{p+1}(x_s^i - q - 1) + q + 1.$$

Similarly, since $F_p(x_s^i - t)$ are initial elements in Z_k for all $t = 0, \ldots, q$, we have

$$F_p(x_s^i) = F_p(x_s^i - q - 1) + q + 1.$$

From Property P4) we deduce that

$$\begin{aligned}
&F_{p+2}(x_s^i - k + 1) - F_{p+1}(x_s^i) \\
&\quad = F_{p+2}(x_s^i - k + 1) - F_{p+1}(x_s^i - q - 1) - q - 1 \\
&\quad > F_{p+1}(x_s^i - k + 1) - F_p(x_s^i - q - 1) - q - 1 \\
&\quad = F_{p+1}(x_s^i - k + 1) - F_p(x_s^i) > 1.
\end{aligned}$$

Hence, (8) holds for $p+1$ and again, by induction, it holds for all $p = 0, 1, \dots, m-1$. From Lemma 2(a) we conclude that

$$F_p(x_s^i) < x_{p+1}^{k+1} < F_{p+1}(x_s^i - k + 1) \quad \text{for each} \quad p \geq 0.$$

Hence, the family of sequences $S_j, j = 1, 2, \dots, k+1$, verify the properties P1) and P2). Moreover, we have $x_1^{k+1} > x_1^k$ for each $k \in \mathbb{N}$ and, since all natural numbers less than x_1^{k+1} belong to Z_k, we conclude that the family of sequences $S_k, k \in \mathbb{N}$ is a disjoint covering of $\mathbb{N}$.

□

For the case $m \geq 2$ and $a_1 = 0$ we do not yet have an answer.

References

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Technical University of Cluj-Napoca, str. Daicoviciu nr. 15, 400020 Cluj-Napoca, Romania

E-mail address: e-mail: Neculae.Vornicescu@math.utcluj.ro