ON HIGHER ORDER LUCAS-BERNOULLI NUMBERS

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Abstract. In this note we consider higher order Bernoulli numbers associated to the formal group laws whose canonical invariant differentials generate the Lucas sequences $\{U_n\}$. We first give an explicit formula for these numbers which implies new identities involving the usual higher order Bernoulli numbers and the Lucas sequences $\{U_n\}$ and $\{V_n\}$. We then give an analogue of the Kummer congruences for these sequences which for each prime p depends only on U_p.

1. Introduction

Let P and Q be integers and consider a Lucas sequence $\{U_n\}$ defined by

$$U_n = PU_{n-1} - QU_{n-2} \quad (n > 1), \qquad U_0 = 0, \quad U_1 = 1. \tag{1.1}$$

Define a power series $\lambda \in \mathbb{Q}[[t]]$ by

$$\lambda(t) = \sum_{n=1}^{\infty} U_n \frac{t^n}{n}. \tag{1.2}$$

Let ε denote the formal compositional inverse of λ in $\mathbb{Q}[[t]]$, and define the *Lucas-Bernoulli numbers* $\hat{B}_n^{(w)}$ *of order* w by the generating function

$$\left(\frac{t}{\varepsilon(t)}\right)^w = \sum_{n=0}^{\infty} \hat{B}_n^{(w)} \frac{t^n}{n!}. \tag{1.3}$$

If one takes $P = -1$ and $Q = 0$ then $U_n = (-1)^{n+1}$ for $n > 0$, $\lambda(t) = \log(1+t)$, $\varepsilon(t) = e^t - 1$, and the numbers $\hat{B}_n^{(w)}$ are the (usual) Bernoulli numbers of order w, denoted simply by $B_n^{(w)}$. The first part of this note centers around an explicit formula for the numbers $\hat{B}_n^{(w)}$ in terms of $B_n^{(w)}$. This formula implies new identities among the sequences $B_n^{(w)}$, U_n, and the companion sequence V_n. In the second part, we prove an analogue of the Kummer congruences for the sequences $\hat{B}_n^{(w)}$. This is an extension of congruences which were proved in the case $P = -1, Q = 0$ in [4] and in the case $w = 1$ in [5].

The power series λ in (1.2) is the formal logarithm of a rational formal group law over $\mathbb{Z}$ (cf. [5], Section 5). In general if one takes λ to be the logarithm of an arbitrary formal group law in characteristic zero then the numbers $\hat{B}_n^{(w)}$ defined by (1.3) are the wth order Bernoulli numbers associated to that formal group law according to the definition in [3]. The Kummer congruences we present in Section 3 for $\hat{B}_n^{(w)}$ depend on the same special element U_p as do those proved in ([5], Theorem 3.2) for $\hat{B}_n^{(1)}$ and have the same modulus as those proved in ([4], Theorem 5.4) for the numbers $\hat{B}_n^{(w)} = B_n^{(w)}$ obtained in (1.3) from the choice $P = -1$, $Q = 0$; in this case the associated formal group law is the multiplicative group law $F(X,Y) = X + Y + XY$. As discussed in ([5], Section 5) in the case $w = 1$, we interpret the strength of our congruences in Section 3 as an expression of the fact that the associated formal group laws are defined over $\mathbb{Z}$, rather than just over $\mathbb{Q}$.

2. Identities for Higher Order Lucas-Bernoulli Numbers

Given integers P and Q we define the Lucas sequence $\{U_n\}$ as in (1.1) and its companion sequence $\{V_n\}$ by

$$V_n = PV_{n-1} - QV_{n-2} \quad (n > 1), \qquad V_0 = 2, \quad V_1 = P. \tag{2.1}$$

Then $r(t) = 1 - Pt + Qt^2$ is the characteristic polynomial of the recurrence for either $\{U_n\}$ or $\{V_n\}$, with discriminant $D = P^2 - 4Q$. If $r(t)$ factors as $r(t) = (1-\alpha t)(1-\beta t)$ then $\alpha = (P+\sqrt{D})/2$ and $\beta = (P-\sqrt{D})/2$, so that $\alpha - \beta = \sqrt{D}$, and for all n we have

$$V_n = \alpha^n + \beta^n, \qquad U_n = \frac{1}{\sqrt{D}}(\alpha^n - \beta^n), \tag{2.2}$$

unless $D = 0$, in which case $U_n = n\alpha^{n-1}$. It follows from (2.2) that

$$\alpha^n = \frac{V_n + U_n\sqrt{D}}{2} \tag{2.3}$$

for all n. For any given Lucas sequence $\{U_n\}$ as in (1.1) we define the numbers $\hat{B}_n^{(w)}$ for $n \geq 0$ by (1.3), and we define $\hat{B}_n^{(w)} = 0$ for $n < 0$.

Theorem 1. *Let $\hat{B}_n^{(w)}$ denote the numbers defined in (1.3). Then for all $m \geq 0$,*

$$\frac{\hat{B}_m^{(w)}}{m!} = \sum_{k=0}^{m} \binom{w}{k} \alpha^k \sqrt{D}^{m-k} \frac{B_{m-k}^{(w-k)}}{(m-k)!}.$$

If $D = 0$ this reduces to

$$\frac{\hat{B}_m^{(w)}}{m!} = \binom{w}{m}\alpha^m.$$

Proof. From ([5], equation (3.4)) we have

$$\frac{t}{\varepsilon(t)} = \alpha t + \frac{\sqrt{D}t}{e^{\sqrt{D}t} - 1} \tag{2.4}$$

so that

$$\left(\frac{t}{\varepsilon(t)}\right)^w = \sum_{k=0}^{\infty} \binom{w}{k} (\alpha t)^k \left(\frac{\sqrt{D}t}{e^{\sqrt{D}t}-1}\right)^{w-k}. \tag{2.5}$$

The $P = -1$, $Q = 0$ case of (1.3) reads

$$\left(\frac{t}{e^t - 1}\right)^w = \sum_{n=0}^{\infty} B_n^{(w)} \frac{t^n}{n!}, \tag{2.6}$$

so from (1.3) and (2.5) we obtain

$$\sum_{m=0}^{\infty} \hat{B}_m^{(w)} \frac{t^m}{m!} = \sum_{k=0}^{\infty} \binom{w}{k} (\alpha t)^k \sum_{s=0}^{\infty} B_s^{(w-k)} \frac{(\sqrt{D}t)^s}{s!} \tag{2.7}$$

and equating coefficients of t^m gives the statement of the theorem; the summation runs from $k = 0$ to m since $B_{m-k}^{(w-k)} = 0$ in the case $k > m$. In the case $D = 0$ (2.4) becomes

$$\frac{t}{\varepsilon(t)} = \alpha t + 1 \tag{2.8}$$

and therefore (2.7) becomes

$$\sum_{m=0}^{\infty} \hat{B}_m^{(w)} \frac{t^m}{m!} = \sum_{k=0}^{\infty} \binom{w}{k} (\alpha t)^k, \tag{2.9}$$

so that $\hat{B}_m^{(w)}/m! = \binom{w}{m}\alpha^m$ when $D = 0$, completing the proof. □

We define

$$\lambda(k) = \begin{cases} V_k, & \text{if } k \text{ is even,} \\ U_k, & \text{if } k \text{ is odd,} \end{cases} \qquad \eta(k) = \begin{cases} U_k, & \text{if } k \text{ is even,} \\ V_k, & \text{if } k \text{ is odd,} \end{cases} \tag{2.10}$$

and restate Theorem 1 as follows.

Corollary. *Let $\hat{B}_n^{(w)}$ denote the numbers defined in (1.3). If $D \neq 0$, then for all $m \geq 0$,*

$$\begin{aligned} \frac{\hat{B}_m^{(w)}}{m!} &= \frac{1}{2} D^{m/2} \sum_{k=0}^{m} \binom{w}{k} \lambda(k) D^{-[k/2]} \frac{B_{m-k}^{(w-k)}}{(m-k)!} \\ &\quad + \frac{1}{2} D^{(m+1)/2} \sum_{k=0}^{m} \binom{w}{k} \eta(k) D^{-[(k+1)/2]} \frac{B_{m-k}^{(w-k)}}{(m-k)!}. \end{aligned}$$

Proof. Substitute (2.3) into Theorem 1 to obtain

$$\frac{\hat{B}_m^{(w)}}{m!} = \sum_{k=0}^{m} \binom{w}{k} \left(\frac{V_k \sqrt{D}^{m-k} + U_k \sqrt{D}^{m+1-k}}{2} \right) \frac{B_{m-k}^{(w-k)}}{(m-k)!}. \tag{2.11}$$

Collecting the terms in (2.11) whose power of $\sqrt{D}$ has the same parity as m, and those of opposite parity, gives the statement of the corollary. □

Remarks. In this theorem and corollary the order w may be taken to lie in any commutative ring with unity. However, if w is taken to be a rational number then each sum in this corollary consists of rational terms. If in addition P, Q are chosen so that the discriminant D is not a square we may then obtain identities for these sums by virtue of the fact that $\hat{B}_m^{(w)}$ is rational. In particular, if m is even then

$$\sum_{k=0}^{m} \binom{w}{k} \eta(k) D^{-[(k+1)/2]} \frac{B_{m-k}^{(w-k)}}{(m-k)!} = 0 \tag{2.12}$$

and

$$\frac{\hat{B}_m^{(w)}}{m!} = \frac{1}{2} D^{m/2} \sum_{k=0}^{m} \binom{w}{k} \lambda(k) D^{-[k/2]} \frac{B_{m-k}^{(w-k)}}{(m-k)!}. \tag{2.13}$$

Conversely if m is odd then

$$\sum_{k=0}^{m} \binom{w}{k} \lambda(k) D^{-[k/2]} \frac{B_{m-k}^{(w-k)}}{(m-k)!} = 0 \tag{2.14}$$

and

$$\frac{\hat{B}_m^{(w)}}{m!} = \frac{1}{2} D^{(m+1)/2} \sum_{k=0}^{m} \binom{w}{k} \eta(k) D^{-[(k+1)/2]} \frac{B_{m-k}^{(w-k)}}{(m-k)!}. \tag{2.15}$$

The identities (2.12) and (2.14) seem to be new identities for the usual higher order Bernoulli numbers.

3. Congruences for Higher Order Lucas-Bernoulli Numbers

For the remainder of this paper we regard the order w as a positive integer. Let p denote an odd prime, $\mathbb{Z}_p$ the ring of p-adic integers, $\mathbb{Q}_p$ the field of p-adic numbers, and $\mathbb{Z}_{(p)}$ the ring of rational numbers with denominator relatively prime to p, so that $\mathbb{Z}_{(p)} = \mathbb{Z}_p \bigcap \mathbb{Q}$. We denote by "ord" the additive valuation on $\mathbb{Q}_p$ defined so that $\operatorname{ord} x = k$ if $p^{-k}x$ is a unit in $\mathbb{Z}_p$. The Pochhammer symbol (or rising factorial) is defined by $(m+1)_w = (m+w)!/m!$. For a sequence $\{a_m\}$ and a nonnegative integer c, we define the action of the forward difference operator Δ_c with increment c by

$$\Delta_c\, a_m = a_{m+c} - a_m. \tag{3.1}$$

The powers Δ_c^k of Δ_c are defined by $\Delta_c^0 =$ identity and $\Delta_c^k = \Delta_c \circ \Delta_c^{k-1}$ for positive integers k, so that

$$\Delta_c^k\, a_m = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} a_{m+jc} \tag{3.2}$$

for all nonnegative integers k. We will have need of the identity

$$\Delta_c^k\{X_m Y_m\} = \sum_{i=0}^{k} \binom{k}{i} \Delta_c^i\{X_m\} \Delta_c^{k-i}\{Y_{m+ic}\}, \tag{3.3}$$

which was observed in ([4], equation (5.38)).

As in Section 5 of [4], for a given nonnegative integer m and a positive integer w we define

$$J = J(m,w) = \{j \in \{1,2,...,w\} \,:\, p-1 \,|\, m+j\}; \tag{3.4}$$

$$M = M(m,w) = \max_{j\in J} \{1 + \operatorname{ord}(m+j)\}; \tag{3.5}$$

$$E = E(m,w) = \sum_{j \in J \cup \{w\}} k(j,m,w), \tag{3.6}$$

$$\text{where} \quad k(j,m,w) = \begin{cases} \max\{1 + \operatorname{ord}(m+j) - \operatorname{ord} j, 0\}, & \text{if } j \in J \text{ and } j \neq w, \\ 1 + \operatorname{ord}(m+j) - \operatorname{ord} j, & \text{if } j = w \in J, \\ -\operatorname{ord} j, & \text{if } j = w \notin J. \end{cases} \tag{3.7}$$

By definition we set $M = 0$ if J is empty. We recall that if $0 \le m \le n$ and $m \equiv n \pmod{(p-1)p^a}$ for some $a \ge M$, then $E(m,w) = E(n,w)$. In ([4], Theorem 5.1) we observe that

$$\operatorname{ord} \frac{B_{m+w}^{(w)}}{(m+1)_w} \ge -E. \tag{3.8}$$

We also observe from equations (5.6) and (5.16) of [4] that

$$E(m,w) \ge E(m,w-s) - \operatorname{ord} \binom{w}{s} \tag{3.9}$$

for $0 \le s \le w$.

Theorem 2. *Let $\hat{B}_n^{(w)}$ denote the numbers defined in (1.3). Then if p is an odd prime and $c = l(p-1)$ where p^a divides l for some $a \geq M$, then for all $m, w, k \geq 0$, the congruence*

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} U_p^{(k-j)l} \frac{\hat{B}^{(w)}_{m+w+jc}}{(m+jc+1)_w} \equiv 0 \qquad (\mathrm{mod}\, p^C \mathbb{Z}_{(p)})$$

holds, where $C = \min\{m - E, k(a+1-M) - E\}$.

Proof. Begin by replacing m with $m+w$ in Theorem 1 and multiplying both sides by $m!$ to obtain

$$\frac{\hat{B}^{(w)}_{m+w}}{(m+1)_w} = \sum_{s=0}^{w} \binom{w}{s} \alpha^s \sqrt{D}^{m+w-s} \frac{B^{(w-s)}_{m+w-s}}{(m+1)_{w-s}}. \tag{3.10}$$

Taking $c = l(p-1)$ as described, the left side of the congruence of the theorem may be expressed via (3.10) as

$$\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} U_p^{(k-j)l} \frac{\hat{B}^{(w)}_{m+w+jc}}{(m+jc+1)_w} \tag{3.11}$$
$$= \sum_{s=0}^{w} \binom{w}{s} \alpha^s \sqrt{D}^{w-s} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} U_p^{(k-j)l} \sqrt{D}^{m+jc} \frac{B^{(w-s)}_{m+w-s+jc}}{(m+jc+1)_{w-s}}.$$

Suppose first that p divides D; then p also divides U_p by ([5], equation (2.4)). The p-adic ordinal of the term indexed by s and j in the sum (3.11) is therefore at least

$$\mathrm{ord}\binom{w}{s} + \frac{m+jc+w-s}{2} + (k-j)l - E(m, w-s) \tag{3.12}$$

since $E(m+jc, w-s) = E(m, w-s)$ for all j. Since $c = l(p-1)$ with $l \geq p^a \geq a+1$ this ordinal is at least

$$\mathrm{ord}\binom{w}{s} + kl + \frac{jl(p-3)}{2} + \frac{m+w-s}{2} - E(m, w-s)$$
$$\geq k(a+1) - E(m, w) \geq C \tag{3.13}$$

which proves the theorem in the case where p divides D.

Now suppose that p does not divide D. We use (3.2) and (3.3) to rewrite the sum in (3.11) as

$$\sum_{s=0}^{w} \binom{w}{s} \alpha^s \sqrt{D}^{w-s} U_p^{kl+\frac{m}{p-1}} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \left(U_p^{-\frac{1}{p-1}} \sqrt{D}\right)^{m+jc} \frac{B^{(w-s)}_{m+w-s+jc}}{(m+jc+1)_{w-s}}$$
$$= \sum_{s=0}^{w} \binom{w}{s} \alpha^s \sqrt{D}^{w-s} U_p^{kl+\frac{m}{p-1}} \Delta_c^k \left\{ \left(U_p^{-\frac{1}{p-1}} \sqrt{D}\right)^{m} \frac{B^{(w-s)}_{m+w-s}}{(m+1)_{w-s}} \right\} \tag{3.14}$$
$$= \sum_{s=0}^{w} \binom{w}{s} \alpha^s \sqrt{D}^{w-s} U_p^{kl+\frac{m}{p-1}} \sum_{i=0}^{k} \binom{k}{i} \Delta_c^i \left\{ \frac{B^{(w-s)}_{m+w-s}}{(m+1)_{w-s}} \right\} \Delta_c^{k-i} \left\{ \left(U_p^{-\frac{1}{p-1}} \sqrt{D}\right)^{m+ic} \right\}.$$

As in ([5], equation (3.8)) we have

$$U_p^{kl+m/(p-1)}\Delta_c^{k-i}\left\{\left(U_p^{-1/(p-1)}\sqrt{D}\right)^{m+ic}\right\}$$
$$=\sqrt{D}^{m+ic}U_p^{(k-i)l}\left(\left(\frac{D^{(p-1)/2}}{U_p}\right)^l-1\right)^{k-i}. \tag{3.15}$$

Since $D^{(p-1)/2} \equiv U_p \pmod p$ by ([5], equation (2.4)), we have $(D^{(p-1)/2}/U_p)^l \equiv 1 \pmod{p^{(a+1)}\mathbb{Z}_{(p)}}$, and therefore (3.15) is zero modulo $p^{(k-i)(a+1)}\mathbb{Z}_{(p)}$. By ([4], Theorem 5.4), we also have

$$\Delta_c^i\left\{\frac{B_{m+w-s}^{(w-s)}}{(m+1)_{w-s}}\right\} \equiv 0 \pmod{p^{C_i}\mathbb{Z}_p} \tag{3.16}$$

where $C_i = \min\{m - E(m, w-s), i(a+1-M(m, w-s)) - E(m, w-s)\}$. Therefore,

$$\binom{w}{s}\Delta_c^i\left\{\frac{B_{m+w-s}^{(w-s)}}{(m+1)_{w-s}}\right\} \equiv 0 \pmod{p^{C_i'}\mathbb{Z}_p} \tag{3.17}$$

where $C_i' = \min\{m - E(m, w), i(a+1-M(m, w)) - E(m, w)\}$. It follows that each term in the last sum of (3.14) is zero modulo $p^C\mathbb{Z}_p$ with C as in the statement of the theorem. This completes the proof. □

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