PARTIAL FRACTION EXPANSIONS AND A QUESTION OF BRUCKMAN

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Abstract. This paper provides a combinatorial proof of an identity which arose in the work of Paul Bruckman. This identity is given by Equation (1.4). We then proceed to generalize Bruckman's Identity via Lagrange Interpolation.

1. Introduction

In this paper we study some interesting binomial identities arising from partial fraction expansions. Some of the formulas are old and some apparently new. Our remarks are inspired by a question raised by Paul Bruckman [1] who posed the following problem:

Bruckman's Problem: Show that

$$\frac{8}{2N+1}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k+n}\binom{2k-2}{k-1}\binom{k+N}{2k}=\frac{4}{2n+1}, \tag{1.1}$$

is valid for $n = 1, 2, ..., N$.

We first rephrase the problem by using the binomial identity

$$\binom{k+N}{2k}\binom{2k}{k}=\binom{k+N}{k}\binom{N}{k}, \tag{1.2}$$

so that Equation (1.1) may be rewritten in the form

$$\sum_{k=1}^{N}(-1)^{k-1}\binom{N}{k}\binom{N+k}{k}\frac{k}{(2k-1)(k+n)}=\frac{2N+1}{2n+1}, \tag{1.3}$$

valid for $n = 1, 2, ..., N$.

With a slight change in notation, let x be a real or complex number other than $-1, -2, ..., -n$. Then, Bruckman's question can be proposed as follows.

Restatement of Bruckman's Problem: Show that

$$\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{n+k}{k}\frac{k}{(2k-1)(k+x)}=\frac{2n+1}{2x+1}, \tag{1.4}$$

is equivalent to the equation $(x-1)(x-2)...(x-n) = 0$.

The purpose of this paper is to verify Equation (1.4). This verification is done in Section 2. We then use Section 3 to discuss a generalization of Equation (1.4). The existence of this generalization relies on the partial fraction decomposition of the Lagrange Interpolation Theorem.

2. Proof of Bruckman's Problem

Our proof of Equation (1.4) utilizes the creative telescoping techniques of Wilf and Zeilberger [3]. In order to make use of creative telescoping, we analyzed the left hand sum in Equation (1.4) for small values of n. By inspecting the calculations results, we formed the following conjecture.

Conjecture 2.1.

$$(2x+1)\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{n+k}{k}\frac{k}{(2k-1)(k+x)}$$
$$= (2n+1)-(-1)^{n}\frac{(x-1)(x-2)...(x-n)}{(x+1)(x+2)...(x+n)}. \tag{2.1}$$

Thus, if we could prove Conjecture 2.1 we will also have proven Equation (1.4).

In order to prove Conjecture 2.1, we use Zeilberger's Algorithm (Chapter 6 of [3]), to find a recurrence relation for the summand term $F(n,k)=(-1)^{k-1}\binom{n}{k}\binom{n+k}{k}\frac{k}{(2k-1)(k+x)}$. By using the ct command in the Ekhad Maple package [3], we find that $F(n,k)$ obeys the following second order recurrence, namely

$$\begin{aligned}&(n+2)(x-n-1)F(n,k)+(2n^2+x+6n+4)F(n+1,k)\\&-(n+1)(x+n+2)F(n+2,k)=G(n,k+1)-G(n,k),\end{aligned} \tag{2.2}$$

where

$$G(n,k)=\frac{2k(k-1)(2k-1)(x+k)}{(n+2-k)(n+1-k)}F(n,k). \tag{2.3}$$

By summing both sides of Equation (2.2) with respect to k, we find that

$$\begin{aligned}&(n+2)(x-n-1)f(n)+(2n^2+x+6n+4)f(n+1)\\&-(n+1)(x+n+2)f(n+2)=0,\end{aligned} \tag{2.4}$$

where

$$f(n)=(2x+1)\sum_{k}F(n,k).$$

Note that $f(n)$ is the left hand side of Equation (2.1). Thus, if we can show that the right side of Equation (2.1) also obey the second order recurrence given by Equation (2.4), then by uniqueness of recurrence solution, we will have proven Conjecture 2.1 and thus, have proven Equation (1.4).

Fortunately, it is an easy computer algebra exericse in Maple to show that $(2n+1)-(-1)^{n}\frac{(x-1)(x-2)...(x-n)}{(x+1)(x+2)...(x+n)}$ satifies Equation (2.4). Therefore, we know Conjecture 2.1 is true via creative telescoping.

3. Generalization of Bruckman's Problem

Bruckman [1] expressed curiosity as to what to do with the $2k-1$ in the denominator of Equation (1.3). As a matter of fact the $2k-1, 2n+1$ and $2x+1$ in Equation (1.4) may be made to disappear in a way that makes it possible to conjecture even more. Noting that

$$\frac{2}{2k-1}=\frac{1}{-\frac{1}{2}+k} \qquad \frac{2x+1}{2}=x-(-\frac{1}{2}) \qquad 2n+1=\frac{\binom{n+\frac{1}{2}}{n}}{\binom{n-\frac{1}{2}}{n}} \tag{3.1}$$

we rewrite Equation (1.4) in the form

$$\left(x-(-\frac{1}{2})\right)\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{n+k}{k}\frac{k}{-\frac{1}{2}+k}\frac{1}{x+k}=\frac{\binom{n-(-\frac{1}{2})}{n}}{\binom{n+(-\frac{1}{2})}{n}}-\frac{\binom{n-x}{n}}{\binom{n+x}{n}}. \tag{3.2}$$

One is then tempted to replace $-\frac{1}{2}$ by an arbitrary number y and assert the conjecture that

$$(x-y)\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{n+k}{k}\frac{k}{x+k}\frac{1}{y+k}=\frac{\binom{n-x}{n}}{\binom{n+x}{n}}-\frac{\binom{n-y}{n}}{\binom{n+y}{n}} \tag{3.3}$$

for all $n \geq 1$ and arbitrary complex x and y not equal to $-1, -2, ..., -n$.

This turns out to be a true consequence of a partial fraction expansion coming from the Lagrange Interpolation Theorem.

Remark 3.1. *Use of relations (3.1), (3.2), and (3.3) is an old device that has been used by Gould for many similar partial fraction examples.*

A well-known partial fraction expansion ([2], Equation (Z.12)) is

$$\frac{f(k)}{\prod_{i=1}^{m}(k+u_i)}=\sum_{j=1}^{m}\frac{f(-u_j)}{\prod^{\alpha}_{\substack{i=1\\ i\neq j}}(u_i-u_j)}\frac{1}{k+u_j} \tag{3.4}$$

where f(x) is a polynomial of degree $\leq m+1$. Partial fraction expansions of this type are derived in Schwatt [4].

Using this we find

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{n+k}{k}\frac{f(k)}{\prod_{i=1}^{m}(k+u_i)}\\ &=\sum_{j=1}^{m}\frac{f(-u_j)}{\prod^{\alpha}_{\substack{i=1\\ i\neq j}}(u_i-u_j)}\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{n+k}{k}\frac{1}{k+u_j}.\end{aligned} \tag{3.5}$$

The inner summation here may be found by setting y = 0 in Equation (3.3), which yields

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{n+k}{k}\frac{1}{k+x}=\frac{1}{x}\frac{\binom{n-x}{n}}{\binom{n+x}{n}}. \tag{3.6}$$

Thus, we find using this with Equation (3.5) that

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{n+k}{k}\frac{f(k)}{\prod_{i=1}^{m}(k+u_i)}=\sum_{j=1}^{m}\frac{f(-u_j)}{\prod^{\alpha}_{\substack{i=1\\ i\neq j}}(u_i-u_j)}\frac{1}{u_j}\frac{\binom{n-u_j}{n}}{\binom{n+i_j}{n}}. \tag{3.7}$$

This then is an m-term generalization of the basic series considered by Bruckman.

References

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[3] M. Petkovsek, H. Wilf, and D. Zeilberger, $A = B$, Academic Press, 1997.

[4] I. J. Schwatt, *Introduction to the Operations with Series*, Univ. of Pennsylvania Press, 1924; Corrected reprint by Chelsea Publ., N.Y. 1962.

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