SYMMETRIC RATIONAL EXPRESSIONS IN THE FIBONACCI NUMBERS

MARTIN GRIFFITHS

Abstract. In this paper we consider the evaluation of numerical expressions obtained by specializing the n variables of a symmetric rational function to the Fibonacci numbers. In particular, we derive both exact and asymptotic formulas for elementary symmetric expressions in the Fibonacci numbers, and go on to demonstrate some applications of these results. The asymptotic formula is then generalized to all sequences sharing a particular mathematical property with the Fibonacci sequence.

1. Introduction

A *symmetric rational function* in the variables $x_1, x_2, \ldots, x_n$ is left unchanged by any permutation of these variables (unchanged, that is, other than in the order of the terms and factors). To take an example,

$$f(x_1, x_2, x_3) = \frac{x_1^2 x_2}{x_3} + \frac{x_1 x_2^2}{x_3} + \frac{x_1^2 x_3}{x_2} + \frac{x_1 x_3^2}{x_2} + \frac{x_2^2 x_3}{x_1} + \frac{x_2 x_3^2}{x_1}$$

is a symmetric rational function in x_1, x_2, and x_3. The *elementary symmetric polynomial* $e_{k,n}$ is defined as the sum of all possible products of k distinct elements from the set $\{x_1, x_2, \ldots, x_n\}$. The elementary symmetric polynomials in three variables are thus given by $e_{1,3} = x_1 + x_2 + x_3$, $e_{2,3} = x_1x_2 + x_1x_3 + x_2x_3$ and $e_{3,3} = x_1x_2x_3$.

A well-known result is that any symmetric rational function in n variables can always be expressed as a rational function in $e_{1,n}, e_{2,n}, \ldots, e_{n,n}$ (for a proof of this see Theorem 13.5.1 in [2]). For example,

$$f(x_1, x_2, x_3) = \frac{e_{1,3}e_{2,3}^2 - e_{2,3,}e_{3,3} - 2e_{1,3}^2 e_{3,3}}{e_{3,3}}.$$

We may therefore think of the elementary symmetric polynomials as basic building blocks for symmetric rational functions.

In this article the variables $x_1, x_2, \ldots, x_n$ are first specialized to the Fibonacci numbers by setting $x_k = F_k$, $k = 1, 2, \ldots, n$. We use $S_{k,n}$ to denote the *elementary symmetric Fibonacci expression* consisting of the sum of all possible products of k elements from $\{F_1, F_2, \ldots, F_n\}$ having distinct indices (if $k > n$ then $S_{k,n}$ is defined to be zero). We obtain exact formulas for some elementary symmetric Fibonacci expressions and, for any fixed $k \in \mathbb{N}$, a general asymptotic formula for $S_{k,n}$. We then go on to illustrate some applications of our results. Finally, our asymptotic formula is generalized further to cater for all Fibonacci-like sequences.

2. Exact formulas

It is not hard to prove the recursive formula

$$e_{k,n} = e_{k,n-1} + x_n e_{k-1,n-1},$$

which, for the Fibonacci numbers, specializes to $S_{k,n} = S_{k,n-1} + F_n S_{k-1,n-1}$. Thus, noting that

$$S_{1,n} = \sum_{k=1}^{n} F_k = F_{n+2} - 1,$$

we have, for $n \geq 3$,

$$S_{2,n} = S_{2,n-1} + F_n S_{1,n-1} = S_{2,n-1} + F_n(F_{n+1} - 1),$$

from which it can be seen that

$$\begin{aligned} S_{2,n} &= \left(\sum_{k=1}^{n} F_k F_{k+1} - F_2 F_3 \right) - \left(\sum_{k=1}^{n} F_k - F_1 - F_2 \right) \\ &= \sum_{k=1}^{n} F_k (F_{k+1} - 1) \\ &= F_n F_{n+2} - F_{n+2} + \frac{1}{2}\left(1 + (-1)^n\right) \\ &= \frac{1}{2}\left(F_{n+1}^2 + F_n F_{n+2} - 2F_{n+2} + 1\right). \end{aligned}$$

Next, for $n \geq 4$,

$$\begin{aligned} S_{3,n} &= S_{3,n-1} + F_n S_{2,n-1} \\ &= S_{3,n-1} + \frac{F_n}{2}\left(F_n^2 + F_{n-1}F_{n+1} - 2F_{n+1} + 1\right). \end{aligned}$$

From this it follows that

$$\begin{aligned} S_{3,n} &= \frac{1}{10}\left(F_{3n+2} - (-1)^n F_{n-1}\right) - \frac{1}{2}\left(F_{n+1}^2 + F_n F_{n+2} - F_{n+2}\right) \\ &= \frac{1}{10}\left(F_{3n+2} + F_n F_{2n} - F_{n+1}F_{2n-1}\right) - \frac{1}{2}\left(F_{n+1}^2 + F_n F_{n+2} - F_{n+2}\right). \end{aligned}$$

In a similar manner it is possible to derive formulas for $S_{4,n}$, $S_{5,n}$, and so on.

3. Asymptotic formulas

It can be verified that

$$S_{1,n} \sim \frac{\phi^{n+2}}{\sqrt{5}},\ S_{2,n} \sim \frac{\phi^{2n+2}}{5},\ S_{3,n} \sim \frac{\phi^{3n+2}}{10\sqrt{5}},\ S_{4,n} \sim \frac{\phi^{4n+2}}{25(5+\sqrt{5})}, \ldots,$$

where

$$\phi = \frac{\sqrt{5}+1}{2} = \frac{1}{\phi - 1} \tag{3.1}$$

is the golden ratio. More generally we find, for fixed k, that

$$S_{k,n} \sim \frac{\phi^{kn+2}}{d_k},$$

where d_k satisfies the recurrence relation

$$d_k = \frac{\sqrt{5}\left(\phi^k - 1\right)}{\phi} d_{k-1} \tag{3.2}$$

for $k \geq 2$, with $d_1 = \sqrt{5}$.

On using (3.1) and (3.2) we have:

$$S_{1,n} \sim \frac{\phi^{n+1}}{\sqrt{5}(\phi - 1)}, \quad S_{2,n} \sim \frac{\phi^{2n+2}}{5(\phi-1)(\phi^2-1)}, \quad \ldots,$$

leading to the following explicit asymptotic formula for any fixed $k \in \mathbb{N}$:

$$\begin{aligned} S_{k,n} &\sim \frac{\phi^{k(n+1)}}{\left(\sqrt{5}\right)^k (\phi-1)(\phi^2-1)\ldots(\phi^k-1)} \\ &= \frac{\phi^{\frac{k}{2}(2n-k+1)}}{\left(\sqrt{5}\right)^k (1-\frac{1}{\phi})(1-\frac{1}{\phi^2})\ldots(1-\frac{1}{\phi^k})} \\ &= \frac{\phi^{\frac{k}{2}(2n-k+1)}}{\left(\sqrt{5}\right)^k} P_k\left(\frac{1}{\phi}\right), \end{aligned} \tag{3.3}$$

where

$$P_k(x) = \prod_{m=1}^{k} \frac{1}{1-x^m}$$

is the generating function for $p_l(k)$, the number of partitions of l into parts not exceeding k (see [1], for example).

4. Some applications

We can use the results from Sections 2 and 3 to derive formulas for symmetric rational expressions in the Fibonacci numbers, and to tackle related problems:

(a) As a first example, let us consider

$$\sum F_k^2 F_m,$$

where the sum is taken over all distinct ordered pairs (k, m) from the set $\{1, 2, \ldots, n\}$. It may be noted that of the $\frac{1}{2}n^2(n-1)$ terms in the expansion of $S_{1,n}S_{2,n}$, $\frac{1}{2}n(n-1)(n-2)$ are of the form $F_kF_lF_m$, where k, l and m are distinct elements from $\{1, 2, \ldots, n\}$, while the remaining $n(n-1)$ terms are of the form $F_k^2F_m$. For a particular choice of three distinct integers, k, l and m say, from $\{1, 2, \ldots, n\}$, there are exactly three appearances of the term $F_kF_lF_m$ in the expansion of $S_{1,n}S_{2,n}$. On the other hand, each term of the form $F_k^2F_m$ appears in the expansion exactly once. From this we see that

$$\begin{aligned}\sum F_k^2 F_m &= S_{1,n}S_{2,n} - 3S_{3,n}\\ &= \frac{1}{2}\left(F_{n+2}-1\right)\left(F_{n+1}^2 + F_nF_{n+2} - 2F_{n+2} + 1\right)\\ &\quad - \frac{3}{10}\left(F_{3n+2} + F_nF_{2n} - F_{n+1}F_{2n-1}\right)\\ &\quad + \frac{3}{2}\left(F_{n+1}^2 + F_nF_{n+2} - F_{n+2}\right)\\ &= \frac{1}{2}\left(F_{n+1}^2F_{n+2} + F_nF_{n+2}^2 - 2F_nF_{n+1} - 1\right)\\ &\quad - \frac{3}{10}\left(F_{3n+2} + F_nF_{2n} - F_{n+1}F_{2n-1}\right).\end{aligned}$$

(b) Next we obtain an asymptotic formula for

$$\left(\sum \frac{1}{F_{k_1}F_{k_2}\dots F_{k_{n-3}}}\right)^{-1},$$

where the sum is taken over all possible sets of $n-3$ distinct elements, $\{k_1, k_2, \dots, k_{n-3}\}$, from $\{1, 2, \dots, n\}$. This expression may be rewritten as

$$\left(\sum \frac{F_kF_lF_m}{F_1F_2\dots F_n}\right)^{-1} = \frac{F_1F_2\dots F_n}{S_{3,n}},$$

where the sum on the left now ranges over all distinct trios, k, l and m, from $\{1, 2, \dots, n\}$. An asymptotic formula for the numerator is given by

$$\frac{C\phi^{\frac{n(n+1)}{2}}}{\left(\sqrt{5}\right)^n},$$

where C is the *Fibonacci factorial constant* defined as

$$C = \prod_{k=1}^{\infty}\left(1 - \left(-\frac{1}{\phi^2}\right)^k\right) = 1.226742\dots$$

(see, for example, [3] and [4]). We therefore have

$$\begin{aligned}\left(\sum \frac{1}{F_{k_1}F_{k_2}\dots F_{k_{n-3}}}\right)^{-1} &\sim \frac{10\sqrt{5}}{\phi^{3n+2}} \times \frac{C\phi^{\frac{n(n+1)}{2}}}{\left(\sqrt{5}\right)^n}\\ &= \frac{2C\phi^{\frac{n^2-5n-4}{2}}}{\left(\sqrt{5}\right)^{n-3}}.\end{aligned}$$

(c) With $g_n(x_n) = (1 + F_1x_n)(1 + F_2x_n)\dots(1 + F_nx_n)$, our results can be used to obtain good numerical approximations to $\lim_{n\to\infty} g_n(x_n)$ for various sequences $\{x_n\}$. Let us, for example, consider the evaluation of

$$\lim_{n\to\infty} g_n\left(\frac{1}{F_n}\right).$$

In order to facilitate this calculation we may note both that

$$g_n(x_n) = 1 + S_{1,n}x_n + S_{2,n}x_n^2 + \dots + S_{n,n}x_n^n$$

and, for any $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{S_{k,n}}{F_n^k} = \frac{\phi^{\frac{k}{2}(2n-k+1)}}{\left(\sqrt{5}\right)^k} P_k\left(\frac{1}{\phi}\right) \times \frac{\left(\sqrt{5}\right)^k}{\phi^{kn}}$$
$$= \phi^{\frac{k(1-k)}{2}} P_k\left(\frac{1}{\phi}\right).$$

The resulting series, given by

$$1 + P_1\left(\frac{1}{\phi}\right) + \frac{1}{\phi} P_2\left(\frac{1}{\phi}\right) + \frac{1}{\phi^3} P_3\left(\frac{1}{\phi}\right) + \ldots,$$

converges quite quickly to $\lim_{n\to\infty} g_n\left(\frac{1}{F_n}\right)$. Indeed, by the tenth term the relative error is less than 1 in 2×10^9.

(d) It is also interesting to note that the constants C and $P_k\left(\frac{1}{\phi}\right)$ appearing in this and the previous section are related in the sense that they may both be expressed in terms of generating functions of partition functions whose arguments are simple functions of ϕ. We have

$$\frac{1}{C} = P\left(-\frac{1}{\phi^2}\right) = \lim_{k\to\infty} P_k\left(-\frac{1}{\phi^2}\right),$$

where $P_k(x)$ is the function defined previously, and $P(x)$ is the generating function for the unrestricted partition function. Although it does need to be borne in mind that our asymptotic formulas are only valid for fixed k, we note that $\lim_{k\to\infty} P_k\left(\frac{1}{\phi}\right)$ also exists. Indeed,

$$\lim_{k\to\infty} P_k\left(\frac{1}{\phi}\right) = P\left(\frac{1}{\phi}\right) = 8.278013\ldots,$$

providing us with the incidental result that, within an order of magnitude, $S_{k,n}$ is given by

$$\frac{\phi^{\frac{k}{2}(2n-k+1)}}{\left(\sqrt{5}\right)^{k-2}}.$$

5. Generalizing

Let us define a sequence of positive integers, $\{G_n\}$ as follows. With $a, b \in \mathbb{N}$, set $G_1 = a$ and $G_2 = b$. Now, for $n \geq 3$, we define G_n recursively as $G_n = G_{n-2} + G_{n-1}$. Thus $\{G_n\}$ is a Fibonacci-like sequence with, as is easily verified,

$$G_n = aF_{n-2} + bF_{n-1}$$
$$= \frac{1}{\sqrt{5}}\left(\phi^{n-2}(a + b\phi) - \left(-\frac{1}{\phi}\right)^{n-2}\left(a - \frac{b}{\phi}\right)\right).$$

Using $S_{k,n}(a, b)$ to denote the sum of all possible products of k elements from $\{G_1, G_2, \ldots, G_n\}$ having distinct indices, we may obtain, in a manner analogous to that used to derive (3.3), the asymptotic formula

$$S_{k,n}(a, b) \sim \left(\frac{a + b\phi}{\sqrt{5}}\right)^k \phi^{\frac{k}{2}(2n-k-3)} P_k\left(\frac{1}{\phi}\right)$$

for fixed k.

It is possible to take the generalization of (3.3) still further. Let $c, d \in \mathbb{R}$ with $c > 0$ and $d > 1$. Suppose the sequence $\{H_n\}$ has the following property:

$$|H_n - cd^n| \to 0 \quad \text{as} \quad n \to \infty.$$

Then, for fixed k,

$$T_{k,n}(c,d) \sim \left(cd^{\frac{1}{2}(2n-k+1)}\right)^k P_k\left(\frac{1}{d}\right),$$

where $T_{k,n}(c,d)$ represents the sum of all possible products of k elements from $\{H_1, H_2, \ldots, H_n\}$ with distinct indices.

References

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Department of Mathematical Sciences, University of Essex, Colchester, CO4 3SQ, United Kingdom

E-mail address: griffm@essex.ac.uk