SOME CONGRUENCES INVOLVING EULER NUMBERS

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Abstract. In this paper, we obtain some explicit congruences for Euler numbers modulo an odd prime power in an elementary way.

1. Introduction

The classical *Bernoulli polynomials* $B_n(x)$ and *Euler polynomials* $E_n(x)$ are usually defined by the exponential generating functions:

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!} \quad \text{and} \quad \frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

The rational numbers $B_n=B_n(0)$ and integers $E_n=2^nE_n(1/2)$ are called *Bernoulli numbers* and *Euler numbers*, respectively. Here are some well-known identities of $B_n(x)$ and $E_n(x)$ (see [11]):

$$B_n(1-x)=(-1)^nB_n(x), \quad B_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}(y)x^k, \tag{1.1}$$

$$E_n(1-x)=(-1)^nE_n(x), \quad E_n(x+y)=\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(y)x^k. \tag{1.2}$$

In particular,

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_{n-k}x^k, \quad E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}, \tag{1.3}$$

and

$$B_n(x+1)-B_n(x)=nx^{n-1}, \quad E_n(x+1)+E_n(x)=2x^n. \tag{1.4}$$

Meanwhile, there exists a close connection between Bernoulli polynomials and Euler polynomials that can be expressed in the following way:

$$E_n(x)=\frac{2^{n+1}}{n+1}\left(B_{n+1}\left(\frac{x+1}{2}\right)-B_{n+1}\left(\frac{x}{2}\right)\right). \tag{1.5}$$

Bernoulli and Euler numbers and polynomials are of particular importance in number theory because they have connections with p-adic analysis and ideal class groups of cyclotomic fields (for example [9], p. 100–109 and [13], p. 29–86). It is also very fascinating and quite useful to investigate arithmetic properties of these numbers and polynomials. For the work in this area the interested readers may consult [2]. Here we give two classical results (see [4], p. 233–240 or [12]).

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Kummer's congruences. Let p be an odd prime and n a positive integer. Then
(1) $E_{(p-1)+2n} \equiv E_{2n} \pmod{p}$.
(2) If $p-1 \nmid 2n$ then

$$\frac{B_{(p-1)+2n}}{(p-1)+2n} \equiv \frac{B_{2n}}{2n} \pmod{p}.$$

von Standt-Clausen Theorem. If n is a positive integer, then

$$B_{2n} + \sum_{p-1|2n} \frac{1}{p} \quad \text{is an integer,}$$

where the sum is over all primes p such that $p-1 \mid 2n$.

Recently, some researchers considered the congruences for Euler numbers, and obtained some beautiful results. For example, let p be an odd prime, Zhang [15] showed that

$$E_{p-1} \equiv 1 + (-1)^{(p+1)/2} \pmod{p}. \tag{1.6}$$

In 2002, Wagstaff [12] gave a more general result: let p be an odd prime and a a positive integer, then $E_n \equiv 0$ or $2 \pmod{p^{a+1}}$ according to $p \equiv 1$ or $3 \pmod 4$ where n is a positive integer such that $(p-1)p^a \mid n$. Wagstaff's proof depends on the result of Johnson [6]: $e_p(p^m/m!) > (p-2)m/(p-1)$ where p is a prime, m is a positive integer, and $e_p(n) = r$ means $p^r \mid n$ but $p^{r+1} \nmid n$. In 2004, Chen [1] derived that

$$E_{k\phi(p^a)+2n} \equiv \big(1 - (-1)^{(p-1)/2} p^{2n}\big) E_{2n} \pmod{p^a}, \tag{1.7}$$

where k is a positive integer, n is a non-negative integer, p^a is an odd prime power with $a \geqslant 1$, and $\phi(n)$ is the Euler function. In 2008, Jakubec [5] established a beautiful connection between Euler numbers and Fermat quotients, where the Euler numbers satisfy that for any prime p with $p \equiv 1 \pmod 4$,

$$E_{p-1} \equiv 0 \pmod{p} \quad \text{and} \quad 2E_{p-1} \equiv E_{2p-2} \pmod{p^2}. \tag{1.8}$$

In this paper, using an elementary way, we obtain some explicit congruences for Euler numbers modulo an odd prime power. From now on we always let $\{x\}$ be the fractional part of x. For a given prime p, $\mathbb{Z}_p$ denotes the set of rational p-integers (those rational numbers whose denominators are not divisible by p). If $x_1, x_2 \in \mathbb{Z}_p$ and $x_1 - x_2 \in p^n\mathbb{Z}_p$, then we say that x_1 is congruent to x_2 modulo p^n and denote this relation by $x_1 \equiv x_2 \pmod{p^n}$. A good introduction to p-adic numbers can be found in [8].

2. Several Lemmas

We begin with a useful identity involving Bernoulli polynomials.

Lemma 2.1. *Let n and m be positive integers, then for any integers r and k with $k \geqslant 0$ we have*

$$\sum_{\substack{x=0 \\ m|x-r}}^{n-1} x^k = \frac{m^k}{k+1}\left(B_{k+1}\left(\frac{n}{m} + \left\{\frac{r-n}{m}\right\}\right) - B_{k+1}\left(\left\{\frac{r}{m}\right\}\right)\right).$$

Proof. It is easy to see that

$$\begin{aligned}
&B_{k+1}\left(\frac{n}{m}+\left\{\frac{r-n}{m}\right\}\right)-B_{k+1}\left(\left\{\frac{r}{m}\right\}\right)\\
=&\sum_{x=0}^{n-1}\left(B_{k+1}\left(\frac{x+1}{m}+\left\{\frac{r-x-1}{m}\right\}\right)-B_{k+1}\left(\frac{x}{m}+\left\{\frac{r-x}{m}\right\}\right)\right)\\
=&\begin{cases}\sum_{x=0}^{n-1}\left(B_{k+1}\left(\frac{x}{m}+\left\{\frac{r-x}{m}\right\}\right)-B_{k+1}\left(\frac{x}{m}+\left\{\frac{r-x}{m}\right\}\right)\right)=0, & \text{if } m\nmid x-r;\\ \sum_{x=0}^{n-1}\left(B_{k+1}\left(\frac{x}{m}+1\right)-B_{k+1}\left(\frac{x}{m}\right)\right), & \text{if } m\mid x-r.\end{cases}
\end{aligned}$$

Thus, by (1.4) we can easily deduce the result of Lemma 2.1. □

The case $m = 1$ in Lemma 2.1 is a well-known fact (see [4], p. 231). The consideration to establish the relation in Lemma 2.1 stems from Lemma 3.1 of Sun [10]. Here, we only consider a special case.

Lemma 2.2. *Let p be a prime and m a positive integer. Then*
(1) $p^m/(m+1)$ is a p-integer, and if $m \geqslant 2$ then $p^m/(m+1) \in p\mathbb{Z}_p$.
(2) pB_m is a p-integer. In particular, if $p-1 \nmid m$ then B_m/m is a p-integer.

Proof. See [4], p. 235–238. □

Lemma 2.3. *Let p be an odd prime, a and k be positive integers. Assume that $x_1, x_2 \in \mathbb{Z}_p$ and $x_1 \equiv x_2 \pmod{p^a}$. If $p-1 \nmid k$ then we have*

$$\frac{B_{k+1}(x_1)}{k+1} \equiv \frac{B_{k+1}(x_2)}{k+1} \pmod{p^a}.$$

Proof. By (1.1), we have

$$\begin{aligned}
\frac{B_{k+1}(x_1)-B_{k+1}(x_2)}{k+1}&=\sum_{r=1}^{k+1}\binom{k}{r-1}B_{k+1-r}(x_2)\frac{(x_1-x_2)^r}{r}\\
=&\sum_{r=1}^{k+1}\binom{k}{r-1}p^{ar-r}pB_{k+1-r}(x_2)\left(\frac{x_1-x_2}{p^a}\right)^r\frac{p^{r-1}}{r}\\
=&\frac{p^akB_k(x_2)}{k}\left(\frac{x_1-x_2}{p^a}\right)\\
&+\sum_{r=2}^{k+1}\binom{k}{r-1}p^{ar-r}pB_{k+1-r}(x_2)\left(\frac{x_1-x_2}{p^a}\right)^r\frac{p^{r-1}}{r}. \qquad (2.1)
\end{aligned}$$

For any non-negative integer m, by (1.1) and Lemma 2.2 we obtain that

$$pB_m(x_2)=\sum_{r=0}^{m}\binom{m}{r}pB_{m-r}x_2^r \in \mathbb{Z}_p.$$

It follows that $\big(B_{k+1}(x_1)-B_{k+1}(x_2)\big)/(k+1) \in \mathbb{Z}_p$. Assume that n is a positive integer such that $n \equiv x_2 \pmod{p}$, then by the fact $\sum_{r=0}^{n-1} r^{k-1} = (B_k(n)-B_k)/k$ we have

$$\frac{B_k(x_2)-B_k}{k}=\frac{B_k(x_2)-B_k(n)}{k}+\frac{B_k(n)-B_k}{k} \in \mathbb{Z}_p.$$

So if $p-1\nmid k$, then by Lemma 2.2 we obtain that $B_k(x_2)/k\in\mathbb{Z}_p$. It follows from (2.1) that if $p-1\nmid k$ then

$$\frac{B_{k+1}(x_1)-B_{k+1}(x_2)}{k+1}\in p^a\mathbb{Z}_p.$$

This completes the proof of Lemma 2.3. □

Lemma 2.4. *Let p be an odd prime, a and k be positive integers. Let m,t be integers with $m\geqslant 1$ and $p\nmid m$. If $p-1\nmid k$ then we have*

$$\sum_{\substack{r \text{ integer}\\ \frac{(t-1)p^a}{m}<r\leqslant\frac{tp^a}{m}}} r^k\equiv\frac{(-1)^k}{k+1}\left(B_{k+1}\left(\left\{\frac{(t-1)p^a}{m}\right\}\right)-B_{k+1}\left(\left\{\frac{tp^a}{m}\right\}\right)\right)\pmod{p^a}.$$

Proof. Observe that

$$\begin{aligned}\sum_{\substack{x=0\\ m\mid x-tp^a}}^{p^a-1} x^k &= \sum_{\substack{r\text{ integer}\\ 0\leqslant tp^a-rm<p^a}}(tp^a-rm)^k=\sum_{\substack{r\text{ integer}\\ \frac{(t-1)p^a}{m}<r\leqslant\frac{tp^a}{m}}}(tp^a-rm)^k\\ &\equiv (-m)^k\sum_{\substack{r\text{ integer}\\ \frac{(t-1)p^a}{m}<r\leqslant\frac{tp^a}{m}}} r^k \pmod{p^a}.\end{aligned}$$

Taking $r=tp^a$ and $n=p^a$ in Lemma 2.1, the result follows from Lemma 2.3. □

Lemma 2.5. *Let m be an odd integer with $m\geqslant 1$. Then for any non-negative integer n we have*

$$E_n\equiv\sum_{l=0}^{m-1}(-1)^l(2l+1)^n\pmod{m}.$$

Proof. Substituting $m+1/2$ for x in (1.3) we have

$$2^nE_n\left(m+\frac{1}{2}\right)=2^n\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}m^{n-k}\equiv E_n\pmod{m}.\tag{2.2}$$

Note that

$$E_n\left(\frac{1}{2}\right)+E_n\left(m+\frac{1}{2}\right)=\sum_{l=0}^{m-1}\left((-1)^lE_n\left(l+\frac{1}{2}\right)-(-1)^{l+1}E_n\left(l+1+\frac{1}{2}\right)\right).$$

It follows from (1.4) that

$$E_n\left(\frac{1}{2}\right)+E_n\left(m+\frac{1}{2}\right)=2\sum_{l=0}^{m-1}(-1)^l\left(l+\frac{1}{2}\right)^n.$$

By the fact $E_n=2^nE_n(1/2)$, we obtain that

$$E_n+2^nE_n\left(m+\frac{1}{2}\right)=2\sum_{l=0}^{m-1}(-1)^l(2l+1)^n.\tag{2.3}$$

Combining (2.2) and (2.3), we have

$$E_n \equiv \sum_{l=0}^{m-1} (-1)^l (2l+1)^n \pmod{m}.$$

Thus, the proof of Lemma 2.5 is completed. □

3. Statement of Results

Since Euler numbers with odd subscripts vanish, $E_{2n+1} = 0$ for all non-negative integer n, it suffices to consider the case E_{2n}. For convenience, in this section we always let $\phi(n)$ be *the Euler function*, and define *the Legendre symbol* $(\frac{m}{p})$, where p is an odd prime and m is any integer, by

$$\left(\frac{m}{p}\right) = \begin{cases} 1, & \text{if } m \text{ is a quadratic residue modulo } p, \\ -1, & \text{if } m \text{ is a quadratic non-residue modulo } p, \\ 0, & \text{if } p \mid m. \end{cases}$$

Theorem 3.1. *Let p be an odd prime with $p \equiv 1 \pmod 4$ and a a positive integer. Then we have*

$$E_{\phi(p^a)/2} \equiv 4 \sum_{r=1}^{\frac{p-1}{4}} \left(\frac{r}{p}\right) \equiv \sum_{l=0}^{p-1} (-1)^l \left(\frac{2l+1}{p}\right) \pmod{p^a}.$$

Proof. Since Bernoulli numbers with odd subscripts vanish, $B_{2n+1} = 0$ for any positive integer n, then taking $m = 4$, $t = 1$ and $k = \phi(p^a)/2$ in Lemma 2.4 we have

$$\sum_{r=1}^{\frac{p^a-1}{4}} r^{\frac{\phi(p^a)}{2}} \equiv \frac{-1}{\phi(p^a)/2+1} B_{\phi(p^a)/2+1}\left(\frac{1}{4}\right) \pmod{p^a}.$$

By (1.5) and (1.1), we obtain

$$E_{2n} = 2^{2n} E_{2n}\left(\frac{1}{2}\right) = -\frac{2^{4n+2}}{2n+1} B_{2n+1}\left(\frac{1}{4}\right).$$

It follows from Fermat's Little Theorem that

$$E_{\phi(p^a)/2} \equiv 4 \sum_{r=1}^{\frac{p^a-1}{4}} r^{\frac{\phi(p^a)}{2}} \pmod{p^a}.$$

By the Euler criterion (see [3], Theorem 83), there exists an integer s such that for any integer r,

$$r^{\frac{p-1}{2}} = sp + \left(\frac{r}{p}\right). \tag{3.1}$$

Thus,

$$r^{\frac{\phi(p^a)}{2}} = \left(sp + \left(\frac{r}{p}\right)\right)^{p^{a-1}} \equiv \left(\frac{r}{p}\right)^{p^{a-1}} \equiv \left(\frac{r}{p}\right) \pmod{p^a}. \tag{3.2}$$

Hence,

$$E_{\phi(p^a)/2} \equiv 4 \sum_{r=1}^{\frac{p^a-1}{4}} \left(\frac{r}{p}\right) \pmod{p^a}. \tag{3.3}$$

On the other hand, taking $n = \phi(p^a)/2$ and $m = p^a$ in Lemma 2.5, then by (3.2) we have

$$E_{\phi(p^a)/2} \equiv \sum_{l=0}^{p^a-1} (-1)^l \left(\frac{2l+1}{p}\right) \pmod{p^a}.$$

By the properties of residue system, it is clear that

$$\sum_{l=0}^{p^a-1} \left(\frac{2l+1}{p}\right) = p^{a-1} \sum_{l=0}^{p-1} \left(\frac{2l+1}{p}\right) = 0.$$

Thus,

$$E_{\phi(p^a)/2} \equiv \sum_{l=0}^{p^a-1} \left((-1)^l - 1\right) \left(\frac{2l+1}{p}\right) = -2 \sum_{l=1}^{\frac{p^a-1}{2}} \left(\frac{4l-1}{p}\right) \pmod{p^a}. \tag{3.4}$$

Note that

$$\sum_{l=1}^{\frac{p^a-1}{2}} \left(\frac{4l-1}{p}\right) = \sum_{l=1}^{\frac{p^a-p}{2}} \left(\frac{4l-1}{p}\right) + \sum_{l=\frac{p^a-p}{2}+1}^{\frac{p^a-1}{2}} \left(\frac{4l-1}{p}\right)$$

$$= \sum_{l=1}^{\frac{p-1}{2}} \left(\frac{4((p^a-p)/2+l)-1}{p}\right) = \sum_{l=1}^{\frac{p-1}{2}} \left(\frac{4l-1}{p}\right) = -\frac{1}{2} \sum_{l=0}^{p-1} (-1)^l \left(\frac{2l+1}{p}\right),$$

and

$$\sum_{r=1}^{\frac{p^a-1}{4}} \left(\frac{r}{p}\right) = \sum_{r=1}^{\frac{p^a-p}{4}} \left(\frac{r}{p}\right) + \sum_{r=\frac{p^a-p}{4}+1}^{\frac{p^a-1}{4}} \left(\frac{r}{p}\right) = \sum_{r=1}^{\frac{p-1}{4}} \left(\frac{(p^a-p)/4+r}{p}\right) = \sum_{r=1}^{\frac{p-1}{4}} \left(\frac{r}{p}\right).$$

The desired result follows immediately from (3.3) and (3.4). □

Remark 3.1. *For a discriminant d let $h(d)$ be the class number of the quadratic field $\mathbb{Q}(\sqrt{d})$ ($\mathbb{Q}$ is the set of rational numbers). If p is a prime of the form $4m+1$, Yuan* [14]*, Lemma 2.3, showed that*

$$2h(-4p) \equiv \sum_{l=0}^{p-1} (-1)^l \left(\frac{2l+1}{p}\right) \not\equiv 0 \pmod{p}.$$

So from Theorem 3.1, we can obtain that $E_{\phi(p^a)/2} \not\equiv 0 \pmod{p^a}$. This gives a different proof of a general conjecture on Euler numbers from Zhang and Xu [16]. Moreover, we also ignore the identity involving Euler numbers (see [7], Lemma 1) which is the key to prove the conjecture by Yuan, Zhang and Xu, respectively.

Remark 3.2. *In* [11]*, Raabe proved a useful theorem that $\sum_{r=0}^{m-1} B_n\big((x+r)/m\big) = m^{1-n} B_n(x)$ for any positive integer m. Taking $x = 0, 1/2$ and $m = 2$ in Raabe's Theorem, it follows from (1.1) that $B_{2n}(3/4) = B_{2n}(1/4) = (1-2^{2n-1})B_{2n}/2^{4n-1}$. If p is a prime such that $p \equiv 3 \pmod{4}$, then in a similar consideration to (3.3) we have*

$$\sum_{r=1}^{\frac{p^a-3}{4}} \left(\frac{r}{p}\right) \equiv \frac{-2B_{\phi(p^a)/2+1}}{\phi(p^a)+2} \left(1 - \frac{(1-2^{\frac{\phi(p^a)}{2}})}{2^{\phi(p^a)+1}}\right) \pmod{p^a}.$$

In particular, if $a = 1$ then by Fermat's Little Theorem we have

$$\sum_{r=1}^{\frac{p-3}{4}} \left(\frac{r}{p}\right) \equiv \left(-1 - \left(\frac{2}{p}\right)\right) B_{(p+1)/2} \pmod{p}.$$

In the same way, we can obtain the Corollary of [4], p. 238,

$$\sum_{r=1}^{\frac{p-1}{2}} \left(\frac{r}{p}\right) \equiv -2\left(2 - \left(\frac{2}{p}\right)\right) B_{(p+1)/2} \pmod{p}.$$

Theorem 3.2. *Let n be a positive integer and p an odd prime. Then*

$$E_{(p-1)+2n} \equiv E_{2n} \pmod{p}.$$

Proof. By Lemma 2.5, we have

$$E_{2n} \equiv \sum_{l=0}^{p-1} (-1)^l (2l+1)^{2n} \pmod{p},$$

and

$$E_{(p-1)+2n} \equiv \sum_{l=0}^{p-1} (-1)^l (2l+1)^{(p-1)+2n} \pmod{p}.$$

It follows from Fermat's Little Theorem that

$$E_{(p-1)+2n} \equiv \sum_{l=0}^{p-1} (-1)^l (2l+1)^{2n} \equiv E_{2n} \pmod{p}.$$

This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let p be an odd prime, a and k be positive integers. Then*

$$E_{k\phi(p^{a+1})} \equiv \begin{cases} 0 \pmod{p^{a+1}}, & \text{if } p \equiv 1 \pmod{4}, \\ 2 \pmod{p^{a+1}}, & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Proof. By Lemma 2.5 and (3.1), we have

$$\begin{aligned} E_{k\phi(p^{a+1})} &= \sum_{l=0}^{p^{a+1}-1} (-1)^l (2l+1)^{k\phi(p^{a+1})} = \sum_{l=0}^{p^{a+1}-1} (-1)^l \left(sp + \left(\frac{2l+1}{p}\right)\right)^{2kp^a} \\ &\equiv \sum_{l=0}^{p^{a+1}-1} (-1)^l \left(\frac{2l+1}{p}\right)^2 \pmod{p^{a+1}}. \end{aligned}$$

Thus,

$$\begin{aligned}
E_{k\phi(p^{a+1})} &\equiv \sum_{l=0}^{\frac{p-3}{2}}(-1)^l + \sum_{l=\frac{p+1}{2}}^{\frac{3p-3}{2}}(-1)^l + \sum_{l=\frac{p+1}{2}+p}^{\frac{3p-3}{2}+p}(-1)^l + \cdots + \sum_{l=\frac{p+1}{2}+p(p^a-2)}^{\frac{3p-3}{2}+p(p^a-2)}(-1)^l \\
&\quad + \sum_{l=\frac{p+1}{2}+p(p^a-1)}^{p-1+p(p^a-1)}(-1)^l = \sum_{l=0}^{\frac{p-3}{2}}(-1)^l + \sum_{l=\frac{p+1}{2}}^{\frac{3p-3}{2}}\Big((-1)^l + (-1)^{p+l} \\
&\quad + (-1)^{2p+l} \cdots + (-1)^{p(p^a-2)+l}\Big) + \sum_{l=\frac{p+1}{2}}^{p-1}(-1)^{p(p^a-1)+l} \\
&= \sum_{l=0}^{\frac{p-3}{2}}(-1)^l + \sum_{l=\frac{p+1}{2}}^{p-1}(-1)^l = \begin{cases} 0, & \text{if } p \equiv 1 \pmod 4, \\ 2, & \text{if } p \equiv 3 \pmod 4. \end{cases}
\end{aligned}$$

This completes the proof of Theorem 3.3. □

We obtain Theorems 3.4 and 3.5 using work from Jakubec [5]. Here we give two more general congruences for Euler numbers.

Theorem 3.4. *Let p be an odd prime, a and k be positive integers. Then*

$$E_{k\phi(p^a)} - kp^{a-1}E_{p-1} \equiv \begin{cases} 0 \pmod{p^{a+1}}, & \text{if } p \equiv 1 \pmod 4, \\ 2 - 2kp^{a-1} \pmod{p^{a+1}}, & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. By Lemma 2.5, we have

$$E_{k\phi(p^a)} \equiv \sum_{l=0}^{p^{a+1}-1}(-1)^l(2l+1)^{k\phi(p^a)} \pmod{p^{a+1}},$$

and

$$E_{p-1} \equiv \sum_{l=0}^{p^{a+1}-1}(-1)^l(2l+1)^{p-1} \pmod{p^{a+1}}.$$

So from (3.1), we obtain

$$\begin{aligned}
&E_{k\phi(p^a)} - kp^{a-1}E_{p-1} \\
&\equiv \sum_{l=0}^{p^{a+1}-1}(-1)^l\left[\left(sp + \left(\frac{2l+1}{p}\right)\right)^{2kp^{a-1}} - kp^{a-1}\left(sp + \left(\frac{2l+1}{p}\right)\right)^2\right] \\
&\equiv (1-kp^{a-1})\sum_{l=0}^{p^{a+1}-1}(-1)^l\left(\frac{2l+1}{p}\right)^2 \pmod{p^{a+1}}. \qquad (3.5)
\end{aligned}$$

By Theorem 3.3, we complete the proof of Theorem 3.4. □

Theorem 3.5. *Let p be an odd prime, a and k be positive integers. Then for any non-negative integer n we have*

$$E_{k\phi(p^a)+2n} - kp^{a-1}E_{p-1+2n} \equiv (1-kp^{a-1})\big(1-(-1)^{\frac{p-1}{2}}p^{2n}\big)E_{2n} \pmod{p^{a+1}}.$$

Proof. For the case $n = 0$, the result is immediate by Theorem 3.4. Now, we consider $n \geqslant 1$. By Lemma 2.5 and (3.5), we have

$$E_{k\phi(p^a)+2n} - kp^{a-1}E_{p-1+2n} \equiv (1 - kp^{a-1}) \sum_{l=0}^{p^{a+1}-1} (-1)^l (2l+1)^{2n} \left(\frac{2l+1}{p}\right)^2$$

$$= (1 - kp^{a-1})\Bigg(\sum_{l=0}^{p^{a+1}-1} (-1)^l (2l+1)^{2n} - p^{2n} \sum_{l=0}^{p^a-1} (-1)^{\frac{p-1}{2}+lp} (2l+1)^{2n} \Bigg)$$

$$= (1 - kp^{a-1})\Bigg(\sum_{l=0}^{p^{a+1}-1} (-1)^l (2l+1)^{2n} - (-1)^{\frac{p-1}{2}} p^{2n} \sum_{l=0}^{p^a-1} (-1)^l (2l+1)^{2n} \Bigg) \pmod{p^{a+1}}.$$

By Lemma 2.5, there exist integers s and t such that

$$\sum_{l=0}^{p^{a+1}-1} (-1)^l (2l+1)^{2n} = E_{2n} + sp^{a+1} \quad \text{and} \quad \sum_{l=0}^{p^a-1} (-1)^l (2l+1)^{2n} = E_{2n} + tp^a.$$

It follows that

$$E_{k\phi(p^a)+2n} - kp^{a-1}E_{p-1+2n} \equiv (1 - kp^{a-1})\big(1 - (-1)^{\frac{p-1}{2}} p^{2n}\big) E_{2n} \pmod{p^{a+1}}.$$

This completes the proof of Theorem 3.5. □

Using a similar proof of Theorem 3.5, we can easily obtain the following theorem.

Theorem 3.6. *Let p be an odd prime, a and k be positive integers. Then for any non-negative integer n we have*

$$E_{k\phi(p^a)+2n} \equiv \big(1 - (-1)^{\frac{p-1}{2}} p^{2n}\big) E_{2n} \pmod{p^a}.$$

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