FIBONACCI IDENTITIES AND GRAPH COLORINGS

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Abstract. We generalize both the Fibonacci and Lucas numbers to the context of graph colorings, and prove some identities involving these numbers. As a corollary we obtain new proofs of some known identities involving Fibonacci numbers such as

$$F_{r+s+t} = F_{r+1}F_{s+1}F_{t+1} + F_rF_sF_t - F_{r-1}F_{s-1}F_{t-1}.$$

1. Introduction

In graph theory, it is natural to study vertex colorings, and more specifically, those colorings in which adjacent vertices have different colors. In this case, the number of such colorings of a graph G is encoded by the chromatic polynomial of G. This object can be computed using the method of "deletion and contraction", which involves the recursive combination of chromatic polynomials for smaller graphs. The purpose of this note is to show how the Fibonacci and Lucas numbers (and other integer recurrences) arise naturally in this context, and in particular, how identities among these numbers can be generated from the different choices for decomposing a graph into smaller pieces.

We first introduce some notation. Let G be a undirected graph (possibly containing loops and multiple edges) with vertices $V = \{1, \ldots, n\}$ and edges E. Given nonnegative integers k and ℓ, a (k, ℓ)-*coloring* of G is a map

$$\varphi \colon V \to \{c_1, \ldots, c_{k+\ell}\},$$

in which $\{c_1, \ldots, c_{k+\ell}\}$ is a fixed set of $k+\ell$ "colors". The map φ is called *proper* if whenever i is adjacent to j and $\varphi(i), \varphi(j) \in \{c_1, \ldots, c_k\}$, we have $\varphi(i) \neq \varphi(j)$. Otherwise, we say that the map φ is *improper*. In somewhat looser terminology, one can think of $\{c_{k+1}, \ldots, c_{k+\ell}\}$ as coloring "wildcards".

Let $\chi_G(x, y)$ be a function such that $\chi_G(k, \ell)$ is the number of proper (k, ℓ)-colorings of G. This object was introduced by the authors of [2] and can be given as a polynomial in x and y (see Lemma 1.1). It simultaneously generalizes the chromatic, independence, and matching polynomials of G. For instance, $\chi_G(x, 0)$ is the usual chromatic polynomial while $\chi_G(x, 1)$ is the independence polynomial for G (see [2] for more details).

We next state a simple rule that enables one to calculate the polynomial $\chi_G(x, y)$ recursively. In what follows, $G\backslash e$ denotes the graph obtained by removing the edge e from G, and for a subgraph H of G, the graph $G\backslash H$ is gotten from G by removing H and all the edges of G that are adjacent to vertices of H. Additionally, the *contraction* of an edge e in G is the graph G/e obtained by removing e and identifying as equal the two vertices sharing this edge.

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Lemma 1.1. *Let e be an edge in G, and let v be the vertex to which e contracts in G/e. Then,*

$$\chi_G(x,y) = \chi_{G\backslash e}(x,y) - \chi_{G/e}(x,y) + y \cdot \chi_{(G/e)\backslash v}(x,y). \tag{1.1}$$

Proof. The number of proper (k,ℓ)-colorings of $G\backslash e$ which have distinct colors for the vertices sharing edge e is given by $\chi_{G\backslash e}(k,\ell) - \chi_{G/e}(k,\ell)$; these colorings are also proper for G. The remaining proper (k,ℓ)-colorings of G are precisely those for which the vertices sharing edge e have the same color. This color must be one of the wildcards $\{c_{k+1},\ldots,c_{k+\ell}\}$, and so the number of remaining proper (k,ℓ)-colorings of G is counted by $\ell \cdot \chi_{(G/e)\backslash v}(k,\ell)$. □

With such a recurrence, we need to specify initial conditions. When G simply consists of one vertex and has no edges, we have $\chi_G(x,y) = x + y$, and when G is the empty graph, we set $\chi_G(x,y) = 1$ (consider G with one edge joining two vertices in (1.1)). Moreover, χ is multiplicative on disconnected components. This allows us to compute χ_G for any graph recursively.

In the special case when $k = 1$, there is also a way to calculate $\chi_G(1,y)$ by removing vertices from G. Define the *link* of a vertex v to be the subgraph $\mathrm{link}(v)$ of G consisting of v, the edges touching v, and the vertices sharing one of these edges with v. Also if u and v are joined by an edge e, we define $\mathrm{link}(e)$ to be $\mathrm{link}(u) \cup \mathrm{link}(v)$ in G, and also we set $\deg(e)$ to be $\deg(u) + \deg(v) - 2$. We then have the following rules.

Lemma 1.2. *Let v be any vertex of G, and let e be any edge. Then,*

$$\chi_G(1,y) = y \cdot \chi_{G\backslash v}(1,y) + y^{\deg(v)} \cdot \chi_{G\backslash \mathrm{link}(v)}(1,y), \tag{1.2}$$

$$\chi_G(1,y) = \quad \chi_{G\backslash e}(1,y) - y^{\deg(e)} \cdot \chi_{G\backslash \mathrm{link}(e)}(1,y). \tag{1.3}$$

Proof. The number of proper $(1,\ell)$-colorings of G with vertex v colored with a wildcard is $\ell \cdot \chi_{G\backslash v}(1,\ell)$. Moreover, in any proper coloring of G with v colored c_1, each vertex among the $\deg(v)$ ones adjacent to v can only be one of the ℓ wildcards. This explains the first equality in the lemma.

Let v be the vertex to which e contracts in G/e. From equation (1.2), we have

$$\chi_{G/e}(1,y) = y \cdot \chi_{(G/e)\backslash v}(1,y) + y^{\deg(v)} \cdot \chi_{(G/e)\backslash \mathrm{link}(v)}(1,y).$$

Subtracting this equation from (1.1) with $x = 1$, and noting that $\deg(e) = \deg(v)$ and $G\backslash \mathrm{link}(e) = (G/e)\backslash \mathrm{link}(v)$, we arrive at the second equality in the lemma. □

Let P_n be the path graph on n vertices and let C_n be the cycle graph, also on n vertices (C_1 is a vertex with a loop attached while C_2 is two vertices joined by two edges). Fixing nonnegative integers k and ℓ, not both zero, we define the following sequences of numbers ($n \geq 1$):

$$\begin{aligned} a_n &= \chi_{P_n}(k,\ell), \\ b_n &= \chi_{C_n}(k,\ell). \end{aligned} \tag{1.4}$$

As we shall see, these numbers are natural generalizations of both the Fibonacci and Lucas numbers to the context of graph colorings. The following lemma uses graph decomposition to give simple recurrences for these sequences.

Lemma 1.3. *The sequences a_n and b_n satisfy the following linear recurrences with initial conditions:*

$$a_1 = k + \ell, \qquad a_2 = (k+\ell)^2 - k, \qquad a_n = (k+\ell-1)a_{n-1} + \ell a_{n-2}; \tag{1.5}$$

$$b_1 = \ell, \qquad b_2 = (k+\ell)^2 - k, \qquad b_3 = a_3 - b_2 + \ell a_1, \tag{1.6}$$

$$b_n = (k+\ell-2)b_{n-1} + (k+2\ell-1)b_{n-2} + \ell b_{n-3}. \tag{1.7}$$

Moreover, the sequence b_n satisfies a shorter recurrence if and only if $k = 0$, $k = 1$, or $\ell = 0$. When $k = 0$, this recurrence is given by $b_n = \ell b_{n-1}$, and when $k = 1$, it is

$$b_n = \ell b_{n-1} + \ell b_{n-2}. \tag{1.8}$$

Proof. The first recurrence follows from deleting an outer edge of the path graph P_n and using Lemma 1.1. To verify the second one, we first use Lemma 1.1 (picking any edge in C_n) to give

$$b_n = a_n - b_{n-1} + \ell a_{n-2}. \tag{1.9}$$

Let $c_n = b_n + b_{n-1} = a_n + \ell a_{n-2}$ and notice that c_n satisfies the same recurrence as a_n; namely,

$$\begin{aligned} c_n &= a_n + \ell a_{n-2} \\ &= (k+\ell-1)a_{n-1} + \ell a_{n-2} + \ell\left((k+\ell-1)a_{n-3} + \ell a_{n-4}\right) \\ &= (k+\ell-1)(a_{n-1} + \ell a_{n-3}) + \ell(a_{n-2} + \ell a_{n-4}) \\ &= (k+\ell-1)c_{n-1} + \ell c_{n-2}. \end{aligned} \tag{1.10}$$

It follows that b_n satisfies the third order recurrence given in the statement of the lemma. Additionally, the initial conditions for both sequences a_n and b_n are easily worked out to be the ones shown.

Finally, suppose that the sequence b_n satisfies a shorter recurrence,

$$b_n + rb_{n-1} + sb_{n-2} = 0,$$

and let

$$B = \begin{bmatrix} b_3 & b_2 & b_1 \\ b_4 & b_3 & b_2 \\ b_5 & b_4 & b_3 \end{bmatrix}.$$

Since the nonzero vector $[1, r, s]^T$ is in the kernel of B, we must have that

$$0 = \det(B) = -k^2(k-1)\ell((k+\ell-1)^2 + 4\ell).$$

It follows that for b_n to satisfy a smaller recurrence, we must have $k = 0$, $k = 1$, or $\ell = 0$. It is clear that when $k = 0$, we have $b_n = \ell^n = \ell b_{n-1}$. When $k = 1$, we can use Lemma 1.2 to see that

$$b_{n+1} = \ell(a_n + \ell a_{n-2}),$$

and combining this with (1.9) gives the recurrence stated in the lemma. □

When $k = 1$ and $\ell = 1$, the recurrences given by Lemma 1.3 when applied to the families of path graphs and cycle graphs are the Fibonacci and Lucas numbers, respectively. This observation is well-known (see [3, Examples 4.1 and 5.3]) and was brought to our attention by Cox [1]:

$$\chi_{P_n}(1,1) = F_{n+2} \qquad \text{and} \qquad \chi_{C_n}(1,1) = L_n. \tag{1.11}$$

Moreover, when $k = 2$ and $\ell = 1$, the recurrence given by Lemma 1.3 when applied to the family of path graphs is the one associated to the Pell numbers:

$$\chi_{P_n}(2,1) = Q_{n+1},$$

where $Q_0 = 1$, $Q_1 = 1$, and $Q_n = 2Q_{n-1} + Q_{n-2}$.

2. Identities

In this section, we derive some identities involving the generalized Fibonacci and Lucas numbers a_n and b_n using the graph coloring interpretation found here. In what follows, we fix $k = 1$. In this case, the a_n and b_n satisfy the following recurrences:

$$a_n = \ell a_{n-1} + \ell a_{n-2} \qquad \text{and} \qquad b_n = \ell b_{n-1} + \ell b_{n-2}.$$

Theorem 2.1. *The following identities hold:*

$$b_n = \ell a_{n-1} + \ell^2 a_{n-3}, \tag{2.1}$$

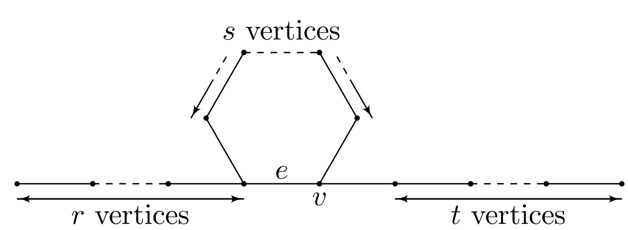
$$b_n = a_n - \ell^2 a_{n-4}, \tag{2.2}$$

$$a_{r+s} = \ell a_r a_{s-1} + \ell^2 a_{r-1} a_{s-2}, \tag{2.3}$$

$$a_{r+s} = a_r a_s - \ell^2 a_{r-2} a_{s-2}, \tag{2.4}$$

$$a_{r+s+t+1} = \ell a_r a_s a_t + \ell^3 a_{r-1} a_{s-1} a_{t-1} - \ell^4 a_{r-2} a_{s-2} a_{t-2}. \tag{2.5}$$

Proof. All the identities in the statement of the theorem follow from Lemma 1.2 when applied to different graphs (with certain choices of vertices and edges). To see the first two equations, consider the cycle graph C_n and pick any vertex and any edge. To see the next two equations, consider the path graph P_{r+s} with $v = r + 1$ and $e = \{r, r + 1\}$.



In order to prove the final equation in the statement of the theorem, consider the graph G in the above figure. It follows from Lemma 1.2 that

$$\ell a_{r+s} a_t + \ell^3 a_{r-1} a_{s-1} a_{t-1} = a_{r+s+t+1} - \ell^4 a_{r-2} a_{s-2} a_{t-1}.$$

Rearranging the terms and applying (2.4), we see that

$$\begin{aligned}
a_{r+s+t+1} &= \ell a_{r+s} a_t + \ell^3 a_{r-1} a_{s-1} a_{t-1} + \ell^4 a_{r-2} a_{s-2} a_{t-1} \\
&= \ell(a_r a_s - \ell^2 a_{r-2} a_{s-2}) a_t + \ell^3 a_{r-1} a_{s-1} a_{t-1} + \ell^4 a_{r-2} a_{s-2} a_{t-1} \\
&= \ell a_r a_s a_t - \ell^3 a_{r-2} a_{s-2} (\ell a_{t-1} + \ell a_{t-2}) \\
&\qquad + \ell^3 a_{r-1} a_{s-1} a_{t-1} + \ell^4 a_{r-2} a_{s-2} a_{t-1} \\
&= \ell a_r a_s a_t + \ell^3 a_{r-1} a_{s-1} a_{t-1} - \ell^4 a_{r-2} a_{s-2} a_{t-2}.
\end{aligned}$$

This completes the proof of the theorem. □

Corollary 2.2. *The following identities hold:*

$$\begin{aligned}
L_n &= F_{n+1} + F_{n-1}, \\
L_n &= F_{n+2} - F_{n-2}, \\
F_{r+s} &= F_{r+1}F_s + F_rF_{s-1}, \\
F_{r+s} &= F_{r+1}F_{s+1} - F_{r-1}F_{s-1}, \\
F_{r+s+t} &= F_{r+1}F_{s+1}F_{t+1} + F_rF_sF_t - F_{r-1}F_{s-1}F_{t-1}.
\end{aligned}$$

Proof. The identities follow from the corresponding ones in Theorem 2.1 with $\ell = 1$ by making suitable shifts of the indices and using (1.11). □

3. Further Exploration

In this note, we have produced recurrences and identities by decomposing different classes of graphs in different ways. Our treatment is by no means exhaustive, and there should be many ways to expand on what we have done here. For instance, is there a graph coloring proof of Cassini's identity?

References

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