ON CONTINUED FRACTION EXPANSIONS OF FIBONACCI AND LUCAS DIRICHLET SERIES

TAKAO KOMATSU

ABSTRACT. We find (non-regular) continued fraction expansions for infinite reciprocal sums of Fibonacci and Lucas numbers. We give continued fraction expansions of some more related series. Moreover, we prove that Fibonacci and Lucas Dirichlet series like $\sum_{n=1}^{\infty} 1/F_n^s$ define hypertranscendental functions, and we investigate the approximates of the series modulo positive integers.

1. INTRODUCTION

In this paper we shall consider the so-called *Fibonacci and Lucas Zeta functions*,

$$\zeta_F(s) = \sum_{n=1}^{\infty} \frac{1}{F_n^s}, \qquad \zeta_L(s) = \sum_{n=0}^{\infty} \frac{1}{L_n^s}.$$

In particular we are interested in the special values for $s = 1$ and $s = 2$, respectively,

$$\zeta_F(1) = \sum_{n=1}^{\infty} \frac{1}{F_n}, \qquad \zeta_F(2) = \sum_{n=1}^{\infty} \frac{1}{F_n^2}, \qquad \zeta_L(1) = \sum_{n=0}^{\infty} \frac{1}{L_n}, \qquad \zeta_L(2) = \sum_{n=0}^{\infty} \frac{1}{L_n^2}.$$

The Fibonacci and Lucas zeta functions are special Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$. Such series typically converge in half-planes $\mathrm{Re}(s) > \sigma_0$ and they can often extend to meromorphic functions on $\mathbb{C}$ like the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Other series, e.g. $\sum p^{-s}$, where p runs over the primes, cannot extend beyond any point on the imaginary axis. So it is natural to ask for the analytic continuation of our Fibonacci and Lucas Zeta functions defined above, for their poles and zeros, and for the residue of their poles. Some interesting work in this direction has been made by L. Navas in [5]. We cite some of his main results. Let $\varphi = (1 + \sqrt{5})/2$.

Proposition 1.1.

- *The Dirichlet series $\sum_{n=1}^{\infty} F_n^{-s}$ can be continued analytically to a meromorphic function on $\mathbb{C}$, whose singularities are simple poles at $s = -2k + \pi i(2n + k)/\log\varphi$, $k \ge 0$, $n \in \mathbb{Z}$, with residue $(-1)^k 5^{s/2}\binom{-s}{k}/\log\varphi$. The series has trivial zeros at $-m$, where $m \ge 0$, $m \equiv 2 \bmod 4$, and the values at other negative integers are rational numbers.*
- *The Dirichlet series $\sum_{n=1}^{\infty} (-1)^n F_n^{-s}$ can be continued analytically to a meromorphic function on $\mathbb{C}$, whose singularities are simple poles at $s = -2k + \pi i(2n + k + 1)/\log\varphi$, $k \ge 0$, $n \in \mathbb{Z}$, with residue $(-1)^k 5^{s/2}\binom{-s}{k}/\log\varphi$. The series has simple poles at $-m$, where $m > 0$, $m \equiv 2 \bmod 4$, and trivial zeros at $-m$, where $m \ge 0$, $m \equiv 0 \bmod 4$.*

This research was partially supported by the Grant-in-Aid for Scientific Research (C) (No. 18540006), the Japan Society for the Promotion of Science.

- *The Dirichlet series* $\sum_{n=1}^{\infty} F_{2n+1}^{-s}$ *can be continued analytically to a meromorphic function on* $\mathbb{C}$*, whose singularities are simple poles at* $s = -2k + \pi in/\log\varphi$*,* $k \geq 0$*,* $n \in \mathbb{Z}$*. Hence, the series has simple poles at all even negative integers. The odd negative integers are trivial zeros of this function.*
- *The Dirichlet series* $\sum_{n=1}^{\infty} F_{2n}^{-s}$ *can be continued analytically to a meromorphic function on* $\mathbb{C}$ *having the same singularities as* $\sum_{n=1}^{\infty} F_{2n+1}^{-s}$ *, and rational values at the odd negative integers.*

Recently, Elsner, Shimomura and Shiokawa showed algebraic dependence and independence results for Fibonacci and Lucas zeta functions. Using Nesterenko's theorem on the Ramanujan functions, it is shown in [1] that the numbers

$$\sum_{n=1}^{\infty}\frac{1}{F_n^2}, \sum_{n=1}^{\infty}\frac{1}{F_n^4}, \sum_{n=1}^{\infty}\frac{1}{F_n^6} \quad \left(\text{respectively, } \sum_{n=1}^{\infty}\frac{1}{L_n^2}, \sum_{n=1}^{\infty}\frac{1}{L_n^4}, \sum_{n=1}^{\infty}\frac{1}{L_n^6}\right)$$

are algebraically independent, and that each

$$\sum_{n=1}^{\infty}\frac{1}{F_n^{2s}} \quad \left(\text{respectively, } \sum_{n=1}^{\infty}\frac{1}{L_n^{2s}}\right) \qquad (s = 4, 5, 6, \dots)$$

is written as a rational (respectively, algebraic) function of these three numbers over $\mathbb{Q}$, e.g.

$$\zeta_F(8) - \frac{15}{14}\zeta_F(4) = \frac{1}{378(4u+5)^2}\Big(256u^6 - 3456u^5 + 2880u^4 + 1792u^3v - 11100u^3 + 20160u^2v - 10125u^2 + 7560uv + 3136v^2 - 1050v\Big),$$

where $u = \zeta_F(2)$ and $v = \zeta_F(6)$. Similar results are obtained in [1] for the alternating sums

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{F_n^{2s}} \quad \left(\text{respectively, } \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{L_n^{2s}}\right) \qquad (s = 1, 2, 3, \dots).$$

From the main theorem in [2] it follows that for any positive distinct integers s_1, s_2, and s_3, the numbers $\zeta_F(2s_1)$, $\zeta_F(2s_2)$, and $\zeta_F(2s_3)$ are algebraically independent if and only if at least one of s_1, s_2, and s_3 is even.

One goal of this paper is to find non-regular continued fraction expansions for $\zeta_F(1)$, $\zeta_L(1)$ and some more extended series, and to compute rational approximations to these numbers from those expansions. We are also interested in the analytical properties of the Fibonacci and Lucas zeta functions as functions in s. We shall prove that these functions are *hypertranscendental*, i.e., they satisfy no algebraic differential equation. Actually, we gain a slightly stronger result by applying Reich's theorem from [7] on Dirichlet series satisfying holomorphic difference - differential equations. Let $\mathcal{D}$ denote the set of all ordinary Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ satisfying the following two conditions:

- The abscissa of absolute convergence is finite: there is some $\sigma_a(f) < \infty$, such that $\sum_{n=1}^{\infty} a_n n^{-s}$ converges absolutely for $s \geq \sigma_a(f)$;
- The set of all divisors of indices n with $a_n \neq 0$ contains infinitely many prime numbers.

Furthermore, we introduce the following notation: For $f \in \mathcal{D}$ and any non-negative integer ν we write

$$\underline{f}^{[\nu]}(s) = \big(f(s), f'(s), \dots, f^{(\nu)}(s)\big).$$

Proposition 1.2. (A. Reich, [7])
Assume that $f \in \mathcal{D}$. *Let* $h_0 < h_1 < \cdots < h_m$ *be any real numbers, and let* $\nu_0, \nu_1, \ldots, \nu_m$ *be any non-negative integers. Let* $\sigma_0 > \sigma_a(f) - h_0$. *Put* $k = m + 1 + \nu_0 + \nu_1 + \cdots + \nu_m$. *If* $\Phi : \mathbb{C}^k \to \mathbb{C}$ *is holomorphic and if the difference-differential equation*

$$\Phi\big(\underline{f}^{[\nu_0]}(s+h_0), \underline{f}^{[\nu_1]}(s+h_1), \ldots, \underline{f}^{[\nu_m]}(s+h_m)\big) = 0$$

holds for all s *with* $\mathrm{Re}(s) > \sigma_0$, *then* Φ *vanishes identically.*

Let $A = (a_n)_{n\geq 1}$ be a bounded sequence of integers. In order to apply Proposition 1.2 to the Fibonacci and Lucas Dirichlet series

$$\zeta_{F,A}(s) := \sum_{n=1}^{\infty} \frac{a_n}{F_n^s} \qquad \text{and} \qquad \zeta_{L,A}(s) := \sum_{n=1}^{\infty} \frac{a_n}{L_n^s} \qquad (a_n \in \mathbb{Z}),$$

respectively, it suffices to show that the sets

$$\begin{aligned} M_F(A) &:= \Big\{p \in \mathbb{P} : \Big(\exists n \geq 1 : a_n \neq 0, p|F_n\Big)\Big\}, \\ M_L(A) &:= \Big\{p \in \mathbb{P} : \Big(\exists n \geq 1 : a_n \neq 0, p|L_n\Big)\Big\} \end{aligned}$$

are not bounded.

Corollary 1. *Let* $A = (a_n)_{n\geq 1}$ *be a sequence with* $a_n \in \{-1, 0, 1\}$ *for* $n \geq 1$ *and* $|M_F(A)| = |M_L(A)| = \infty$. *Then both the functions* $\zeta_{F,A}(s)$ *and* $\zeta_{L,A}(s)$ *are hypertranscendental.*

In section 4 below (see Theorem 4.1) we shall apply this corollary to various sequences $(a_n)_{n\geq 1}$ and to the corresponding Dirichlet series, which have nonregular continued fraction expansions. First, we treat such expansions in the subsequent Sections 2 and 3.

2. Auxiliary Results for Nonregular Continued Fractions

A (nonregular infinite) continued fraction is given by

$$b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots + \cfrac{a_n}{b_n + \ldots}}}}} \tag{2.1}$$

with real numbers a_ν and b_ν. In addition, let $\sum_{\nu\geq 0} c_\nu$ be an absolutely convergent series with $c_\nu \neq 0$. Seidel [8] called such a series and a continued fraction (2.1) *equivalent* if

$$c_0 + c_1 + \cdots + c_\nu = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots + \cfrac{a_\nu}{b_\nu}}}}} \qquad (\nu = 0, 1, \ldots).$$

Define the fraction A_n/B_n $(n = 0, 1, \dots)$ by

$$\frac{A_n}{B_n} = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cfrac{a_4}{\ddots + \cfrac{a_n}{b_n}}}}}. \tag{2.2}$$

From [6, Section 2] we get the well-known recurrence formulas:

$$\begin{array}{llll} A_\nu = b_\nu A_{\nu-1} + a_\nu A_{\nu-2} & (\nu \geq 2), & A_0 = b_0, & A_1 = b_0 b_1 + a_1; \\ B_\nu = b_\nu B_{\nu-1} + a_\nu B_{\nu-2} & (\nu \geq 2), & B_0 = 1, & B_1 = b_1 . \end{array}$$

With Seidel's terminology we use the following Lemma in order to obtain our results on nonregular continued fraction expansions of specific values of Fibonacci and Lucas Dirichlet series (see Section 3).

Lemma 2.1. *The series* $\sum_{\nu \geq 0} c_\nu$ *with* $c_\nu \neq 0$ *for* $\nu \geq 1$ *and the continued fraction*

$$c_0 + \cfrac{c_1}{1 - \cfrac{c_2}{c_1 + c_2 - \cfrac{c_1 c_3}{c_2 + c_3 - \cfrac{c_2 c_4}{c_3 + c_4 - \cfrac{c_3 c_5}{\ddots - \cfrac{c_n c_{n+2}}{c_{n+1} c_{n+2} - \dots}}}}}}$$

are equivalent.

Proof. See Satz 7 and formula (3) in [6, Section 45]. □

Consider generalized Fibonacci numbers $\{G_n\}_{n \geq 1}$ defined by

$$G_n = G_{n-1} + G_{n-2} \quad (n \geq 2)$$

with positive integral initial values G_1 and G_2. Let

$$\zeta_G(s) = \sum_{n=1}^{\infty} \frac{1}{G_n^s}, \qquad \zeta_G^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{G_n^s}.$$

Then we have two Lemmas by using Lemma 2.1.

Lemma 2.2.

$$\zeta_G(s) = \cfrac{1}{G_1^s - \cfrac{G_1^{2s}}{G_1^s + G_2^s - \cfrac{G_2^{2s}}{G_2^s + G_3^s - \cfrac{G_3^{2s}}{G_3^s + G_4^s - \ddots - \cfrac{G_{n-1}^{2s}}{G_{n-1}^s + G_n^s - \dots}}}}}.$$

Proof. By Lemma 2.1 we have

$$\begin{aligned}
\zeta_G(s) &= \cfrac{G_1^{-s}}{1-\cfrac{G_2^{-s}}{G_1^{-s}+G_2^{-s}-\cfrac{G_1^{-s}G_3^{-s}}{G_2^{-s}+G_3^{-s}-\cfrac{G_2^{-s}G_4^{-s}}{G_3^{-s}+G_4^{-s}-\dots}}}} \\
&= \cfrac{1}{G_1^{s}-\cfrac{G_1^{s}G_2^{-s}}{G_1^{-s}+G_2^{-s}-\cfrac{G_1^{-s}G_3^{-s}}{G_2^{-s}+G_3^{-s}-\cfrac{G_2^{-s}G_4^{-s}}{G_3^{-s}+G_4^{-s}-\dots}}}} \\
&= \cfrac{1}{G_1^{s}-\cfrac{G_1^{2s}}{G_1^{s}+G_2^{s}-\cfrac{G_2^{s}G_3^{-s}}{G_2^{-s}+G_3^{-s}-\cfrac{G_2^{-s}G_4^{-s}}{G_3^{-s}+G_4^{-s}-\dots}}}} \\
&= \cfrac{1}{G_1^{s}-\cfrac{G_1^{2s}}{G_1^{s}+G_2^{s}-\cfrac{G_2^{2s}}{G_2^{s}+G_3^{s}-\cfrac{G_3^{s}G_4^{-s}}{G_3^{-s}+G_4^{-s}-\dots}}}} \quad ,
\end{aligned}$$

which implies the identity stated in Lemma 2.2. ◻

Lemma 2.3.

$$\zeta_G^*(s) = \cfrac{1}{G_1^{s}+\cfrac{G_1^{2s}}{-G_1^{s}+G_2^{s}+\cfrac{G_2^{2s}}{-G_2^{s}+G_3^{s}+\cfrac{G_3^{2s}}{-G_3^{s}+G_4^{s}+\ddots+\cfrac{G_{n-1}^{2s}}{-G_{n-1}^{s}+G_n^{s}+\dots}}}}} .$$

Proof. By Lemma 2.1 we have

$$\zeta_G^*(s) = \cfrac{G_1^{-s}}{1 - \cfrac{-G_2^{-s}}{G_1^{-s} - G_2^{-s} - \cfrac{G_1^{-s}G_3^{-s}}{-G_2^{-s} + G_3^{-s} - \cfrac{G_2^{-s}G_4^{-s}}{G_3^{-s} - G_4^{-s} - \dots}}}}$$

$$= \cfrac{1}{G_1^s - \cfrac{-G_1^s G_2^{-s}}{G_1^{-s} - G_2^{-s} - \cfrac{G_1^{-s}G_3^{-s}}{-G_2^{-s} + G_3^{-s} - \cfrac{G_2^{-s}G_4^{-s}}{G_3^{-s} - G_4^{-s} - \dots}}}}$$

$$= \cfrac{1}{G_1^s + \cfrac{G_1^{2s}}{-G_1^s + G_2^s + \cfrac{G_2^s G_3^{-s}}{G_2^{-s} - G_3^{-s} + \cfrac{G_2^{-s}G_4^{-s}}{G_3^{-s} - G_4^{-s} - \dots}}}}$$

$$= \cfrac{1}{G_1^s + \cfrac{G_1^{2s}}{-G_1^s + G_2^s + \cfrac{G_2^{2s}}{-G_2^s + G_3^s + \cfrac{G_3^s G_4^{-s}}{G_3^{-s} - G_4^{-s} - \dots}}}} \,,$$

which implies the identity stated in Lemma 2.3. □

3. Results on Nonregular Continued Fraction Expansions of Fibonacci and Lucas Dirichlet Series

Theorem 3.1. *We have*

$$\zeta_F(1) = \cfrac{1}{F_2 - \cfrac{F_1^2}{F_3 - \cfrac{F_2^2}{F_4 - \cfrac{F_3^2}{\ddots - \cfrac{F_{n-1}^2}{F_{n+1} - \dots}}}}}$$

and

$$\sum_{\nu=1}^{n} \frac{1}{F_\nu} = \frac{A_n}{B_n}\,,$$

where $\{A_\nu\}_{\nu\ge 0}$ *and* $\{B_\nu\}_{\nu\ge 0}$ *are determined by the recurrence formulas:*

$$A_\nu = F_{\nu+1}A_{\nu-1} - F_{\nu-1}^2 A_{\nu-2} \qquad (\nu \ge 2), \qquad A_0 = 0, \qquad A_1 = 1;$$
$$B_\nu = F_{\nu+1}B_{\nu-1} - F_{\nu-1}^2 B_{\nu-2} \qquad (\nu \ge 2), \qquad B_0 = 1, \qquad B_1 = 1\,.$$

Proof. Set $G_n = F_n$ with $F_1 = F_2 = 1$ and $s = 1$ in Lemma 2.2. Since $F_n + F_{n+1} = F_{n+2}$ $(n \geq 1)$ and $F_1 = F_2$ by the definition, we get the result. $\square$

The following table contains 15 continued fraction expansions of certain Fibonacci and Lucas Dirichlet series including the result from Theorem 3.1. In Table 1 we denote the nonregular continued fraction expansion (2.1) of such a series by

$$\sum_{\nu=1}^{\infty} \cdots = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \ddots}}} = a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_3}{a_3} + \cdots .$$

Let the n-th convergent of the series be defined by

$$\sum_{\nu=1}^{n} \cdots = \frac{A_n}{B_n} = a_0 + \frac{b_1}{a_1} + \frac{b_2}{a_2} + \frac{b_3}{a_3} + \cdots + \frac{b_n}{a_n} .$$

Both the sequences $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ satisfy the same linear three term recurrence formula

$$X_\nu = H_\nu X_{\nu-1} + K_\nu X_{\nu-2} \qquad (\nu \geq 2)$$

with initial values $A_0 = 0, A_1 = 1$ and $B_0 = 1, B_1 \in \{1, 3\}$, and with specific functions H_ν, K_ν quoted in Table 1. In Table 2 we refer to the identities used to simplify the denominators of the continued fractions quoted in Table 1.

series	expansion	H_ν	K_ν	B_1
$\sum_{\nu=1}^{\infty} \frac{1}{F_\nu}$	$\frac{1}{F_2 -} \frac{F_1^2}{F_3 -} \cdots - \frac{F_{n-1}^2}{F_{n+1} -} \dots$	$F_{\nu+1}$	$-F_{\nu-1}^2$	1
$\sum_{\nu=1}^{\infty} \frac{1}{F_\nu^2}$	$\frac{1}{F_1 -} \frac{F_1^4}{F_3 -} \cdots - \frac{F_{n-1}^4}{F_{2n-1} -} \dots$	$F_{2\nu-1}$	$-F_{\nu-1}^4$	1
$\sum_{\nu=1}^{\infty} \frac{1}{F_\nu^3}$	$\frac{1}{1 -} \frac{F_1^6}{F_3 + F_0^3 -} \cdots - \frac{F_{n-1}^6}{F_{3n-3} + F_{n-2}^3 -} \dots$	$F_{3\nu-3} + F_{\nu-2}^3$	$-F_{\nu-1}^6$	1
$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{F_\nu}$	$\frac{1}{1 +} \frac{F_1^2}{F_0 +} \cdots + \frac{F_{n-1}^2}{F_{n-2} +} \dots$	$F_{\nu-2}$	$F_{\nu-1}^2$	1
$\sum_{\nu=1}^{\infty} \frac{1}{L_\nu}$	$\frac{1}{1 -} \frac{L_1^2}{L_3 -} \cdots - \frac{L_{n-1}^2}{L_{n+1} -} \dots$	$L_{\nu+1}$	$-L_{\nu-1}^2$	1
$\sum_{\nu=1}^{\infty} \frac{1}{L_\nu^2}$	$\frac{1}{1 -} \frac{L_1^4}{5F_3 -} \cdots - \frac{L_{n-1}^4}{5F_{2n-1} -} \dots$	$5F_{2\nu-1}$	$-L_{\nu-1}^4$	1
$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{L_\nu}$	$\frac{1}{1 +} \frac{L_1^2}{L_0 +} \cdots + \frac{L_{n-1}^2}{L_{n-2} +} \dots$	$L_{\nu-2}$	$L_{\nu-1}^2$	1
$\sum_{\nu=1}^{\infty} \frac{1}{F_{2\nu}}$	$\frac{1}{L_1 -} \frac{F_2^2}{L_3 -} \cdots - \frac{F_{2n-2}^2}{L_{2n-1} -} \dots$	$L_{2\nu-1}$	$-F_{2\nu-2}^2$	1
$\sum_{\nu=1}^{\infty} \frac{1}{L_{2\nu}}$	$\frac{1}{3 -} \frac{L_2^2}{5F_3 -} \cdots - \frac{L_{2n-2}^2}{5F_{2n-1} -} \dots$	$5F_{2\nu-1}$	$-L_{2\nu-2}^2$	3
$\sum_{\nu=1}^{\infty} \frac{1}{F_{2\nu-1}}$	$\frac{1}{1 -} \frac{F_1^2}{L_2 -} \cdots - \frac{F_{2n-3}^2}{L_{2n-2} -} \dots$	$L_{2\nu-2}$	$-F_{2\nu-3}^2$	1
$\sum_{\nu=1}^{\infty} \frac{1}{L_{2\nu-1}}$	$\frac{1}{1 -} \frac{L_1^2}{5F_2 -} \cdots - \frac{L_{2n-3}^2}{5F_{2n-2} -} \dots$	$5F_{2\nu-2}$	$-L_{2\nu-3}^2$	1
$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{F_{2\nu}}$	$\frac{1}{F_1 +} \frac{F_2^2}{F_3 +} \cdots + \frac{F_{2n-2}^2}{F_{2n-1} +} \dots$	$F_{2\nu-1}$	$F_{2\nu-2}^2$	1
$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{L_{2\nu}}$	$\frac{1}{3 +} \frac{L_2^2}{L_3 +} \cdots + \frac{L_{2n-2}^2}{L_{2n-1} +} \dots$	$L_{2\nu-1}$	$L_{2\nu-2}^2$	3
$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{F_{2\nu-1}}$	$\frac{1}{1 +} \frac{F_1^2}{F_2 +} \cdots + \frac{F_{2n-3}^2}{F_{2n-2} +} \dots$	$F_{2\nu-2}$	$F_{2\nu-3}^2$	1
$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu-1}}{L_{2\nu-1}}$	$\frac{1}{1 +} \frac{L_1^2}{L_2 +} \cdots + \frac{L_{2n-3}^2}{L_{2n-2} +} \dots$	$L_{2\nu-2}$	$L_{2\nu-3}^2$	1

Table 1.

series	underlying formula	reference
$\sum_{\nu=1}^{\infty}\frac{1}{F_\nu}$	$F_n + F_{n+1} = F_{n+2}$	
$\sum_{\nu=1}^{\infty}\frac{1}{F_\nu^2}$	$F_n^2 + F_{n+1}^2 = F_{2n+1}$	[4, p.97, formula 30]
$\sum_{\nu=1}^{\infty}\frac{1}{F_\nu^3}$	$F_n^3 + F_{n+1}^3 = F_{3n} + F_{n-1}^3$	[4, p.97, formula 62]
$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{F_\nu}$	$-F_n + F_{n+1} = F_{n-1}$	
$\sum_{\nu=1}^{\infty}\frac{1}{L_\nu}$	$L_n + L_{n+1} = L_{n+2}$	
$\sum_{\nu=1}^{\infty}\frac{1}{L_\nu^2}$	$L_n^2 + L_{n+1}^2 = 5F_{2n+1}$	[4, p.97, formula 37]
$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{L_\nu}$	$-L_n + L_{n+1} = L_{n-1}$	
$\sum_{\nu=1}^{\infty}\frac{1}{F_{2\nu}}$	$F_{2n} + F_{2n+2} = L_{2n+1}$	[4, p.97, formula 32]
$\sum_{\nu=1}^{\infty}\frac{1}{L_{2\nu}}$	$L_{2n} + L_{2n+2} = 5F_{2n+1}$	[4, p.97, formula 34]
$\sum_{\nu=1}^{\infty}\frac{1}{F_{2\nu-1}}$	$F_{2n-1} + F_{2n+1} = L_{2n}$	[4, p.97, formula 32]
$\sum_{\nu=1}^{\infty}\frac{1}{L_{2\nu-1}}$	$L_{2n-1} + L_{2n+1} = 5F_{2n}$	[4, p.97, formula 34]
$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{F_{2\nu}}$	$-F_{2n} + F_{2n+2} = F_{2n+1}$	
$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{L_{2\nu}}$	$-L_{2n} + L_{2n+2} = L_{2n+1}$	
$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{F_{2\nu-1}}$	$-F_{2n-1} + F_{2n+1} = F_{2n}$	
$\sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{L_{2\nu-1}}$	$-L_{2n-1} + L_{2n+1} = L_{2n}$	

Table 2.

4. Hypertranscendence of Fibonacci and Lucas Dirichlet Series

In this section we apply the concept of hypertranscendental functions introduced in Section 1 to our special Dirichlet series.

Theorem 4.1. *The Dirichlet series*

$$\sum_{n=1}^{\infty}\frac{1}{F_n^s}\,,\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_n^s}\,,\quad \sum_{n=1}^{\infty}\frac{1}{L_n^s}\,,\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_n^s}\,,\quad \sum_{n=1}^{\infty}\frac{1}{F_{2n}^s}\,,\quad \sum_{n=1}^{\infty}\frac{1}{L_{2n}^s}\,,$$

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n-1}^s}\,,\quad \sum_{n=1}^{\infty}\frac{1}{L_{2n-1}^s}\,,\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{2n}^s}\,,\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_{2n}^s}\,,\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{F_{2n-1}^s}\,,\quad \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{L_{2n-1}^s}$$

define hypertranscendental functions in s.

Proof. The hypertranscendency of the Dirichlet series $\sum_{n=1}^{\infty} F_n^{-s}$ was proved in [9, Theorem 1] by using the fact that for distinct odd primes p and q, $\gcd(F_p, F_q) = F_{\gcd(p,q)} = F_1 = 1$ [4, Theorem 16.3]. Then the sequence of Fibonacci numbers F_n catches infinitely many primes among their divisors. The same holds for Lucas numbers because for distinct relatively prime positive integers $j \geq 2$ and $k \geq 2$ with same parity, $\gcd(L_j, L_k) = 1$.

Therefore, by Corollary 1 it suffices to show that

$$|M_F(A_\nu)| = \infty \qquad \text{and} \qquad |M_L(B_\nu)| = \infty$$

for $A_1 = (F_{2n})_{n\geq 1}$, $A_2 = (F_{2n-1})_{n\geq 1}$, $B_1 = (L_{2n})_{n\geq 1}$, $B_2 = (L_{2n-1})_{n\geq 1}$. Since for any odd prime p, $F_p | F_{2p}$ [4, Theorem 16.1], we have $|M_F(A_1)| = \infty$. It is clear that $|M_F(A_2)| = \infty$ because $\gcd(F_p, F_q) = F_{\gcd(p,q)}$ again. $|M_L(B_1)| = \infty$ follows from the fact $\gcd(L_{2p}/3, L_{2q}/3) = 1$ for distinct odd primes p, q. It is clear that $|M_L(B_2)| = \infty$ because for any odd distinct primes p, q, $\gcd(L_p, L_q) = 1$. □

5. On the Approximants A_n/B_n mod t

Finally we consider the sequences $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ from (2.2) modulo t for any integer $t \geq 2$. Here, we use a result recently obtained by Elsner and the author [3].

Theorem 5.1. *Let $t \geq 2$ be any integer, and let $\{Y_n\}_{n\geq 0}$ be a sequence of integers satisfying the recurrence relation*

$$Y_\nu = T(\nu)Y_{\nu-1} + U(\nu)Y_{\nu-2} \qquad (\nu \geq 2)$$

with sequences $\{T(\nu)\}_{\nu\geq 2}$ and $\{U(\nu)\}_{\nu\geq 2}$ of integers, which are periodic modulo t. Then the sequence $\{Y_n\}_{n\geq 0}$ is ultimately periodic modulo t. If $U(\nu) = 1$ for all $\nu \geq 2$, then the sequence $\{Y_n\}_{n\geq 0}$ is periodic modulo t.

Since the generalized Fibonacci numbers $\{G_n\}_{n\geq 0}$ satisfy the recurrence relation $G_\nu = G_{\nu-1} + G_{\nu-2}$, the sequence $\{G_n\}_{n\geq 0}$ mod t is periodic for $t \geq 2$. It follows that sequences like

$$\{G_{n+1}\}_{n\geq 2}\,,\quad \{G_{2n-1}\}_{n\geq 2}\,,\quad \{-G_{n-1}^2\}_{n\geq 2}\,,\quad \{-G_{n-1}^4\}_{n\geq 2}\,,\quad \{(-1)^n G_{n-2}\}_{n\geq 2}\,,$$

and so on, are also periodic modulo t. Next, we apply again Theorem 5.1 and the recurrence formulas for the convergents A_n/B_n of our Dirichlet series in order to obtain the following result.

Theorem 5.2. *Let A_n/B_n for $n \geq 0$ be the convergents of the series quoted in Table 1. Then, for any integer $t \geq 2$, the sequences $(A_n)_{n\geq 0}$ and $(B_n)_{n\geq 0}$ are ultimately periodic modulo t.*

The exact periodicity may be different in each case. For example, for A_n and B_n in Theorem 3.1 we have the following.

Theorem 5.3.
(1) *For all* $n \geq 2k$, $A_n \equiv 0 \pmod{F_k}$.
(2) *For all* $n \geq k$, $B_n \equiv 0 \pmod{F_k}$.

Proof. First, by induction we have $B_n = F_1 F_2 \dots F_n$. Thus,

$$A_n = B_n \sum_{\nu=1}^{n} \frac{1}{F_\nu} = F_1 F_2 \dots F_n \sum_{\nu=1}^{n} \frac{1}{F_\nu}\,.$$

It is clear that if $n \geq k$, then

$$B_n = F_1 F_2 \dots F_n \equiv 0 \pmod{F_k}\,.$$

Since $F_n | F_{2n}$ $(n \geq 1)$, we have

$$\frac{F_1 \dots F_k \dots F_{2k} \dots F_n}{F_\nu} \equiv 0 \pmod{F_k} \quad (n \geq 2k,\ 1 \leq \nu \leq n)\,.$$

Hence, if $n \geq 2k$ then

$$A_n = \sum_{\nu=1}^{n} \frac{F_1 F_2 \dots F_n}{F_\nu} \equiv 0 \pmod{F_k}\,.$$

□

6. Acknowledgment

The author is grateful to Professor Jörn Steuding from the University of Würzburg, Germany for giving of his paper [9]. He also thanks Professor Carsten Elsner for useful discussions.

References

[1] C. Elsner, S. Shimomura, and I. Shiokawa, *Algebraic Relations for Reciprocal Sums of Fibonacci Numbers*, Acta Arith., **130** (2007), 37–60.
[2] C. Elsner, S. Shimomura, and I. Shiokawa, *Algebraic Independence Results for Reciprocal Sums of Fibonacci Numbers*, (preprint), `http://www.carstenelsner.de/Abstract48.pdf.`
[3] C. Elsner and T. Komatsu, *On the Residue Classes of Integer Sequences Satisfying a Linear Three Term Recurrence Formula*, Linear Algebra Appl., **429** (2008), 933–947.
[4] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley & Sons, New York, 2001.
[5] L. Navas, *Analytic Continuation of the Fibonacci Dirichlet Series*, The Fibonacci Quarterly, **39** (2001), 409–418.
[6] O. Perron, *Die Lehre von den Kettenbrüchen*, Chelsea Publishing Company, New York, 1950.
[7] A. Reich, *Über Dirichletsche Reihen und holomorphe Differentialgleichungen*, Analysis, **4** (1984), 27–44.
[8] L. Seidel, *Bemerkungen über den Zusammenhang zwischen dem Bildungsgesetze eines Kettenbruches und der Art des Fortgangs seiner Näherungsbrüche*, Abh. Math.-Phys. Classe König. Bayer. Akad. Wiss., **7** (1855), 559–602.
[9] J. Steuding, *The Fibonacci Zeta-Function is Hypertranscendental*, Forschungsberichte der FHDW Hannover, Nr. 02007/06.

MSC2000: 11A55, 11B37, 11B39

Graduate School of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan
E-mail address: `komatsu@cc.hirosaki-u.ac.jp`