THE NUMBER OF FINITE HOMOMORPHISM-HOMOGENEOUS TOURNAMENTS WITH LOOPS

DRAGAN MAŠULOVIĆ

Abstract. A structure is called homogeneous if every isomorphism between finitely induced substructures of the structure extends to an automorphism of the structure. Recently, P. J. Cameron and J. Nešetřil introduced a relaxed version of homogeneity: we say that a structure is homomorphism-homogeneous if every homomorphism between finitely induced substructures of the structure extends to an endomorphism of the structure.

In this short note we compute the number of homomorphism-homogeneous finite tournaments where vertices are allowed to have loops. Our main result is that in case $n \geqslant 4$ there are, up to isomorphism, $F_n + n - 1$ homomorphism-homogeneous tournaments on n vertices, where F_n is the n-th Fibonacci number. This is the only class of homomorphism-homogeneous structures where we can provide an exact number of nonisomorphic objects, and the number turns out to be closely related to Fibonacci numbers.

1. Introduction

A structure is *homogeneous* if every isomorphism between finitely induced substructures of the structure extends to an automorphism of the structure. For example, finite and countably infinite homogeneous tournaments were described in [3].

Theorem 1.1 (Lachlan [3]). *The following are the only finite and countably infinite homogeneous tournaments:*

- *the trivial one-vertex tournament (the tournament with one vertex and no edges),*
- *the oriented 3-cycle,*
- *the transitive tournament $\mathbb{Q}$ (the rationales with the usual order),*
- *the countable dense local order (the set of vertices is a countable dense set of points of the unit circle containing no antipodal pairs, and edges are drawn counterclockwise up to half-way around), and*
- *the homogeneous tournament T containing all finite tournaments (this is the generic tournament associated with the amalgamation class of all finite tournaments).*

In their recent paper [1] the authors discuss a generalization of homogeneity to various types of morphisms between structures, and in particular introduce the notion of homomorphism-homogeneous structures.

Definition 1.2 (Cameron, Nešetřil [1]). *A structure is called* homomorphism-homogeneous *if every homomorphism between finitely induced substructures of the structure extends to an endomorphism of the structure.*

In this short note we compute the number of homomorphism-homogeneous finite tournaments where vertices are allowed to have loops. Our main result is that in case $n \geqslant 4$ there are, up to isomorphism, $F_n + n - 1$ homomorphism-homogeneous tournaments on n vertices,

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where F_n is the n-th Fibonacci number. This is the only class of homomorphism-homogeneous structures where we can provide an exact number of nonisomorphic objects, and the number turns out to be closely related to Fibonacci numbers.

2. Preliminaries

A *digraph* is an ordered pair $D = (V, E)$ where V is a nonempty finite set, the set of *vertices* of D, $E \subseteq V^2$ is a set of *edges* of D and

$$\text{if } (x, y) \in E \text{ and } x \neq y \text{ then } (y, x) \notin E.$$

Edges of the form (x, x) are called *loops*. If $(x, x) \in E$ we also say that x *has a loop*. Instead of $(x, y) \in E$ we often write $x \to y$.

Let $D_1 = (V_1, E_1)$ and $D_2 = (V_2, E_2)$ be digraphs. We say that $f : V_1 \to V_2$ is a *homomorphism* between D_1 and D_2 and write $f : D_1 \to D_2$ if

$$x \to y \text{ implies } f(x) \to f(y), \text{ for all } x, y \in V_1.$$

An *endomorphism* is a homomorphism from D into itself. A mapping $f : V_1 \to V_2$ is an *isomorphism* between D_1 and D_2 if f is bijective and

$$x \to y \text{ if and only if } f(x) \to f(y), \text{ for all } x, y \in V_1.$$

Digraphs D_1 and D_2 are *isomorphic* if there is an isomorphism between them. We write $D_1 \cong D_2$. An *automorphism* is an isomorphism from D onto itself.

A digraph $D' = (V', E')$ is a *subdigraph* of a digraph $D = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. We write $D' \leqslant D$ to denote that D' is isomorphic to a subdigraph of D. For $\varnothing \neq W \subseteq V$ by $D[W]$ we denote the digraph $(W, E \cap W^2)$ which we refer to as the *subdigraph of D induced by W*.

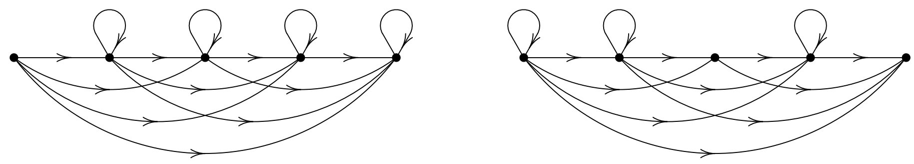
A digraph D is *homogeneous* if every isomorphism $f : W_1 \to W_2$ between finite induced subdigraphs of D extends to an automorphism of D. A digraph D is *homomorphism-homogeneous* if every homomorphism $f : W_1 \to W_2$ between finite induced subdigraphs of D extends to an endomorphism of D (see Definition 1.2).

3. Tournaments

Let $\mathbf{1}$ denote the trivial digraph with only one vertex and no edges, and let $\mathbf{1}^\circ$ denote the digraph with only one vertex with a loop. An *oriented cycle with n vertices* is a digraph C_n whose vertices are 1, 2, ..., n, $n \geqslant 3$, and whose edges are $1 \to 2 \to \ldots \to n \to 1$. By C_n° we denote the digraph obtained from C_n by adding all loops. A digraph D is *acyclic* if $C_n \leqslant D$ for no $n \geqslant 3$. Note that an acyclic digraph may have loops. A digraph $D = (V, E)$ is *transitive* if $x \to y \to z$ implies $x \to z$ for all $x, y, z \in V$.

A *tournament* is a digraph T such that either $x \to y$ or $y \to x$ but not both, whenever x and y are distinct vertices of T. Note that according to our approach, a vertex in a tournament can, but does not have to have a loop. It is well-known that two acyclic tournaments with no loops are isomorphic if and only if they have the same number of vertices. The (up to isomorphism) unique acyclic tournament with no loops on n vertices will be denoted by A_n. By A_n° we denote the acyclic tournament on n vertices where every vertex has a loop. Note also that $A_1 \cong \mathbf{1}$ and $A_1^\circ \cong \mathbf{1}^\circ$.

We shall also need two variations of the notions. Let $A_n^\circ(i)$ denote the acyclic tournament on the set of vertices $\{1, 2, \ldots, n\}$ such that $1 \to 2 \to \ldots \to n$ and where every vertex has a loop except for vertex i; similarly, let $A_n^\circ(i, j)$ denote the acyclic tournament on the set of vertices $\{1, 2, \ldots, n\}$ such that $1 \to 2 \to \ldots \to n$ and where every vertex has a loop except for vertices i and j, see Figure 1.



$A_5^\circ(1)$ $\qquad$ $A_5^\circ(3,5)$

FIGURE 1. Acyclic tournaments $A_5^\circ(1)$ and $A_5^\circ(3,5)$

In [2] finite homomorphism-homogeneous tournaments with loops were described.

Theorem 3.1 ([2])**.** *A finite tournament T is homomorphism-homogeneous if and only if it belongs to one of the following classes:*

(1) $A_n^\circ(1)$ *and* $A_n^\circ(n)$, $n \geqslant 1$;
(2) *an acyclic tournament with two consecutive loopless vertices and with the additional property that both the initial and the final vertex of the tournament have a loop:* $A_n^\circ(i, i+1)$, $1 < i < i+1 < n$;
(3) *a dense acyclic tournament, where an acyclic tournament $T = (V, E)$ is* dense *if the following holds:*
 (i) *there exists a vertex a with a loop such that $a \to x$ for all $x \in V$,*
 (ii) *there exists a vertex b with a loop such that $x \to b$ for all $x \in V$, and*
 (iii) *for any two vertices $x, y \in V$ such that $x \to y$ there exists a vertex $z \in V$ with a loop such that $x \to z \to y$;*
(4) C_3 *or* C_3°.

We can now use the characterization in Theorem 3.1 to compute the number of nonisomorphic finite homomorphism-homogeneous tournaments on n vertices.

Theorem 3.2. *The number of nonisomorphic acyclic dense tournaments on $n \geqslant 1$ vertices is F_n, the n-th Fibonacci number.*

Proof. Since two acyclic tournaments with no loops are isomorphic if and only if they have the same number of vertices, it follows that a dense acyclic tournament on n vertices is uniquely determined by the distribution of its loops. Therefore, if $\{v_1, \ldots, v_n\}$ is the set of vertices of a dense acyclic tournament D where $v_1 \to v_2 \to \ldots \to v_n$, then D is uniquely determined by the string $x_1 \ldots x_n$ of 0's and 1's defined by

$$x_i = \begin{cases} 0, & v_i \text{ does not have a loop} \\ 1, & v_i \text{ has a loop.} \end{cases}$$

The requirement that D be dense translates to the following requirements on the string $x_1 \ldots x_n$:

(i) $x_1 = x_n = 1$, and
(ii) if $x_i = x_j = 0$ and $i < j$, then there is a k such that $i < k < j$ and $x_k = 1$; or, equivalently, there do not exist consecutive zeroes in the string $x_1 \ldots x_n$.

We say that strings of 0's and 1's fulfilling (i) and (ii) are *dense*.

Let $f(n)$ denote the number of nonisomorphic acyclic dense tournaments on n vertices. Then clearly, $f(n)$ equals the number of dense strings of length n. It is now easy to see that $f(1) = f(2) = 1$, and that $f(n) = f(n-1) + f(n-2)$. (Namely, the number of dense strings $x_1 x_2 \dots x_n$ with $x_2 = 1$ is $f(n-1)$ since then $x_2 \dots x_n$ is dense; on the other hand, the number of dense strings $x_1 x_2 \dots x_n$ with $x_2 = 0$ is $f(n-2)$ since then $x_3 = 1$ and $x_3 \dots x_n$ is dense.) Therefore, $f(n) = F_n$. □

Corollary 3.3. *There are two nonisomorphic homomorphism-homogeneous tournaments on one vertex:* **1** *and* $\mathbf{1}^\circ$.

There is, up to isomorphism, only one homomorphism-homogeneous tournament on two vertices: A_2°.

There are, up to isomorphism, six homomorphism-homogeneous tournaments on three vertices.

If $n \geqslant 4$ *then, up to isomorphism, there are* $F_n + n - 1$ *homomorphism-homogeneous tournaments on* n *vertices.*

Proof. The cases $n = 1$ and $n = 2$ are trivial. Assume, therefore, that $n \geqslant 3$ and let

$$\delta_{p,q} = \begin{cases} 1, & p = q \\ 0, & p \neq q. \end{cases}$$

The numbers of nonisomorphic tournaments on n vertices in classes (1), (2) and (4) in Theorem 3.1 are 2, $n - 3$ and $2\delta_{n,3}$, respectively, while from Theorem 3.2 we know that the number of nonisomorphic acyclic dense tournaments on n vertices is F_n. Therefore, if $n \geqslant 3$ then, up to isomorphism, there are $F_n + n + 2\delta_{n,3} - 1$ homomorphism-homogeneous tournaments on n vertices. If $n \geqslant 4$ then $\delta_{n,3} = 0$, and the number of nonisomorphic homomorphism-homogeneous tournaments is $F_n + n - 1$. If, however, $n = 3$ we get that the number of nonisomorphic homomorphism-homogeneous tournaments is $F_3 + 3 + 2 - 1 = 6$. □

References

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Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja, Obradovića 4, 21000 Novi Sad, Serbia

E-mail address: masul@dmi.uns.ac.rs