FINDING g-GONAL NUMBERS IN RECURRENCE SEQUENCES

SZABOLCS TENGELY

ABSTRACT. In this note we find all g-gonal numbers in the Fibonacci, Lucas, Pell and Associated Pell sequences for $g \in \{6, 8, 9, 10, \ldots, 20\}$.

1. INTRODUCTION

The m-th g-gonal number is defined by
$$\mathcal{G}_{m,g} = \frac{m\{(g-2)m - (g-4)\}}{2}.$$
If m is a positive integer and $g = 3, 4, 5, 6, 7, 8, \ldots$, then the number $\mathcal{G}_{m,g}$ is called triangular, square, pentagonal, hexagonal, heptagonal, octagonal, etc. There are many articles concerning the mixed exponential-polynomial Diophantine equation
$$R_n = P(x),$$
where R_n is a linear recursive sequence and $P \in \mathbb{Z}[X]$ is a polynomial. Here we refer to [3, 10, 16, 17, 18, 19, 20] and the references given there. Several papers have been published identifying the numbers $\mathcal{G}_{m,g}$ (for certain values of g) in the Fibonacci sequence $\{F_n\}$, Lucas sequence $\{L_n\}$, Pell sequence $\{P_n\}$ and Associated Pell sequence $\{Q_n\}$, where these sequences are defined by
$$\begin{aligned}
F_0 &= 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2,\\
L_0 &= 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2,\\
P_0 &= 0, \quad P_1 = 1, \quad P_n = 2P_{n-1} + P_{n-2} \text{ for } n \geq 2,\\
Q_0 &= 1, \quad Q_1 = 1, \quad Q_n = 2Q_{n-1} + Q_{n-2} \text{ for } n \geq 2.
\end{aligned}$$
In the table below we summarize related results in cases of $g = 3, 4, 5$ and 7.

	$\mathcal{G}_{m,3}$	$\mathcal{G}_{m,4}$	$\mathcal{G}_{m,5}$	$\mathcal{G}_{m,7}$
F_n	[11]: $\{0,1,3,21,55\}$	[8, 9]: $\{0,1,144\}$	[13]: $\{0,1,5\}$	[26]: $\{0,1,13,34,55\}$
L_n	[12]: $\{1,3\}$	[5]: $\{1,4\}$	[14]: $\{2,1,7\}$	[25]: $\{1,4,7,18\}$
P_n	[15]: $\{0,1\}$	[7]: $\{0,1,169\}$	[23]: $\{0,1,2,5,12,70\}$	[28]: $\{0,1,70\}$
Q_n	[24]: $\{1,3\}$	[7]: $\{1\}$	[22]: $\{1,7\}$	[27]: $\{1,7,99\}$

We note that the result related to the equation $F_n = \mathcal{G}_{m,4}$ is a straightforward consequence of two papers by Ljunggren [8, 9] and it was rediscovered by Cohn [5].

In the present paper we resolve the equations
$$\begin{aligned}
F_n = \mathcal{G}_{m,g}, \quad L_n = \mathcal{G}_{m,g},\\
P_n = \mathcal{G}_{m,g}, \quad Q_n = \mathcal{G}_{m,g},
\end{aligned}$$
for $g \in \{6, 8, 9, 10, \ldots, 20\}$.

Our main result is the following.

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Theorem 1.1. *All the solutions of the equations* $F_n = \mathcal{G}_{m,g}$, $L_n = \mathcal{G}_{m,g}$, $P_n = \mathcal{G}_{m,g}$ *and* $Q_n = \mathcal{G}_{m,g}$ *with* $m, n \geq 0$ *for* $g \in \{6, 8, 9, 10, \ldots, 20\}$ *are those which are summarized in the table below.*

	$\mathcal{G}_{m,6}$	$\mathcal{G}_{m,8}$	$\mathcal{G}_{m,9}$	$\mathcal{G}_{m,10}$	$\mathcal{G}_{m,11}$	$\mathcal{G}_{m,12}$	$\mathcal{G}_{m,13}$
F_n	$\{0,1,3,21,55\}$	$\{0,1,5,8,21,4181\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1,8\}$	$\{0,1\}$	$\{0,1,13\}$
L_n	$\{1,3,5778\}$	$\{1\}$	$\{1\}$	$\{1,7,76\}$	$\{1,11\}$	$\{1\}$	$\{1\}$
P_n	$\{0,1\}$	$\{0,1,5,408\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1,12\}$	$\{0,1,70\}$
Q_n	$\{1,3\}$	$\{1\}$	$\{1\}$	$\{1,7\}$	$\{1\}$	$\{1\}$	$\{1\}$

	$\mathcal{G}_{m,14}$	$\mathcal{G}_{m,15}$	$\mathcal{G}_{m,16}$	$\mathcal{G}_{m,17}$	$\mathcal{G}_{m,18}$	$\mathcal{G}_{m,19}$	$\mathcal{G}_{m,20}$
F_n	$\{0,1,34\}$	$\{0,1\}$	$\{0,1,13\}$	$\{0,1,2584\}$	$\{0,1\}$	$\{0,1,2584\}$	$\{0,1\}$
L_n	$\{1,11,76\}$	$\{1\}$	$\{1\}$	$\{1,322\}$	$\{1,18\}$	$\{1\}$	$\{1\}$
P_n	$\{0,1\}$	$\{0,1,12\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$	$\{0,1\}$
Q_n	$\{1\}$	$\{1\}$	$\{1\}$	$\{1,17,47321\}$	$\{1\}$	$\{1,99\}$	$\{1,17\}$

Proof. The statement follows from Lemma 2, Lemma 3, Lemma 4 and Lemma 5. □

We use the following well-known properties of the sequences F_n, L_n, P_n and Q_n :

$$L_n^2 - 5F_n^2 = 4(-1)^n, \tag{1}$$
$$Q_n^2 - 2P_n^2 = (-1)^n. \tag{2}$$

2. The equation $F_n = \mathcal{G}_{m,g}$

If n is even, then from (1) we get

$$C_{F_n}^{even}: \quad Y^2 = 5((g-2)X^2 - (g-4)X)^2 + 16,$$

and if n is odd, then

$$C_{F_n}^{odd}: \quad Y^2 = 5((g-2)X^2 - (g-4)X)^2 - 16.$$

One can easily check that $(X, Y) = (0, 4)$ is a point on the curve $C_{F_n}^{even}$ and $(X, Y) = (1, 2)$ is a point on $C_{F_n}^{odd}$, that is these curves define elliptic curves. We need to find the integral points on these genus 1 curves for fixed g to solve the equation $F_n = \mathcal{G}_{m,g}$. One can apply the so-called elliptic logarithm method (see [6, 29, 30]). This method is now available in the computer algebra package MAGMA [1]. We have the following result.

Lemma 2.1. *The solutions of the equation* $F_n = \mathcal{G}_{m,g}$ *with* $m, n \geq 0$ *and* $g \in \{6, 8, 9, 10, \ldots, 20\}$ *are*

$$\begin{array}{llll}
F_0 = \mathcal{G}_{0,g}, & F_1 = F_2 = \mathcal{G}_{1,g}, & \textit{where } g \in \{6,8,9,10,\ldots,20\}, & \\
F_4 = \mathcal{G}_{-1,6}, & F_8 = \mathcal{G}_{-3,6}, & F_{10} = \mathcal{G}_{-5,6}, & \\
F_5 = \mathcal{G}_{-1,8}, & F_6 = \mathcal{G}_{2,8}, & F_8 = \mathcal{G}_{3,8}, & F_{19} = \mathcal{G}_{-37,8}, \\
F_6 = \mathcal{G}_{-1,11}, & F_7 = \mathcal{G}_{2,13}, & F_9 = \mathcal{G}_{-2,14}, & F_7 = \mathcal{G}_{-1,16}, \\
F_{18} = \mathcal{G}_{19,17}, & F_{18} = \mathcal{G}_{-17,19}. & &
\end{array}$$

Proof. We have reduced the problem to computing integral points on certain elliptic curves. Using the computer package MAGMA [1], we find the solutions listed in the theorem. □

3. The equation $L_n = \mathcal{G}_{m,g}$

If n is even, then from (1) we get

$$C_{L_n}^{even}: \quad Y^2 = 5((g-2)X^2 - (g-4)X)^2 - 80,$$

and if n is odd, then

$$C_{L_n}^{odd}: \quad Y^2 = 5((g-2)X^2 - (g-4)X)^2 + 80.$$

It is easy to see that $(1,10) \in C_{L_n}^{odd}(\mathbb{Q})$. Here we have the following result.

Lemma 3.1. *The solutions of the equation $L_n = \mathcal{G}_{m,g}$ with $m, n \geq 0$ and $g \in \{6, 8, 9, 10, \ldots, 20\}$ are*

$$\begin{array}{llll} L_1 = \mathcal{G}_{1,g}, & \textit{where } g \in \{6,8,9,10,\ldots,20\}, & & \\ L_2 = \mathcal{G}_{-1,6}, & L_{18} = \mathcal{G}_{54,6}, & L_4 = \mathcal{G}_{-1,10}, & \\ L_9 = \mathcal{G}_{-4,10}, & L_5 = \mathcal{G}_{2,11}, & L_5 = \mathcal{G}_{-1,14}, & L_9 = \mathcal{G}_{4,14}, \\ L_{12} = \mathcal{G}_{7,17}, & L_6 = \mathcal{G}_{2,18}. & & \end{array}$$

Proof. The point $(1,10)$ is on the curve $C_{L_n}^{odd}(\mathbb{Q})$, therefore it is an elliptic curve and we use MAGMA [1] again to find all integral points on these curves. Similarly we have points on $C_{L_n}^{even}(\mathbb{Q})$ for $g \in \{6, 10, 17, 18\}$, so the same method works to solve the problem in these cases. For $g \in \{8, 9, 11, 13, 14, 16, 19\}$ we get that the equations do not have solutions in $\mathbb{Q}_5$. As an example we consider the case $g = 8$. The equation is

$$Y^2 = 180X^4 - 240X^3 + 80X^2 - 80.$$

The above equation does not have solutions modulo 125. It remains to deal with the cases $g \in \{12, 15, 20\}$. The corresponding equations are

$$\begin{aligned} g = 12: &\quad Y^2 = 500X^4 - 800X^3 + 320X^2 - 80, \\ g = 15: &\quad Y^2 = 845X^4 - 1430X^3 + 605X^2 - 80, \\ g = 20: &\quad Y^2 = 1620X^4 - 2880X^3 + 1280X^2 - 80. \end{aligned}$$

We give the proof for $g = 12$; the other two equations can be solved similarly. All solutions of the Diophantine equation $A^2 - 5B^2 = -5C^2$ can be given in parametric form as follows (see e.g. [4]).

$$\begin{aligned} A &= -\frac{t}{h} 10uv, \\ B &= -\frac{t}{h}(u^2 + 5v^2), \\ C &= \frac{t}{h}(u^2 - 5v^2), \end{aligned}$$

where $h \mid 50$ and $\gcd(u,v) = 1$. The equation $Y^2 = 500X^4 - 800X^3 + 320X^2 - 80$ can be written as $Y^2 - 5(10X^2 - 8X)^2 = -5(4)^2$. Hence, we have

$$\begin{aligned} Y &= -\frac{t}{h} 10uv, \\ 10X^2 - 8X &= -\frac{t}{h}(u^2 + 5v^2), \\ 4 &= \frac{t}{h}(u^2 - 5v^2). \end{aligned}$$

Therefore we get that $t \mid 4$. We rewrite the second equation as $(10X-4)^2 - 16 = -10\frac{t}{h}(u^2 + 5v^2)$. Since $16 = 4\frac{t}{h}(u^2 - 5v^2)$ we obtain

$$(10X-4)^2 = -6\frac{t}{h}u^2 - 70\frac{t}{h}v^2.$$

For given t, h we parametrize all solutions of these equations as

$$\begin{aligned} 10X - 4 &= p_{1,t,h}(r,s), \\ u &= p_{2,t,h}(r,s), \\ v &= p_{3,t,h}(r,s), \end{aligned}$$

where $p_{1,t,h}, p_{2,t,h}, p_{3,t,h}$ are homogeneous degree 2 polynomials. Thus we obtain quartic Thue equations

$$4 = \frac{t}{h}(p_{2,t,h}(r,s)^2 - 5p_{3,t,h}(r,s)^2).$$

We use MAGMA [1] to solve these equations and then obtain all solutions of the equation $L_n = \mathcal{G}_{m,12}$. □

4. The equation $P_n = \mathcal{G}_{m,g}$

If n is even, then from (2) we get

$$C_{P_n}^{even}: \quad Y^2 = 2((g-2)X^2 - (g-4)X)^2 + 4,$$

and if n is odd, then

$$C_{P_n}^{odd}: \quad Y^2 = 2((g-2)X^2 - (g-4)X)^2 - 4.$$

We have that $(0,2) \in C_{P_n}^{even}(\mathbb{Q})$ and $(1,2) \in C_{P_n}^{odd}(\mathbb{Q})$. We compute the integral points on these curves and obtain the following result.

Lemma 4.1. *The solutions of the equation $P_n = \mathcal{G}_{m,g}$ with $m, n \geq 0$ and $g \in \{6, 8, 9, 10, \ldots, 20\}$ are*

$$\begin{array}{lll} P_0 = \mathcal{G}_{0,g}, & P_1 = \mathcal{G}_{1,g}, \textit{where } g \in \{6,8,9,10,\ldots,20\}, & \\ P_3 = \mathcal{G}_{-1,8}, & P_8 = \mathcal{G}_{12,8}, & P_4 = \mathcal{G}_{2,12}, \\ P_6 = \mathcal{G}_{4,13}, & P_4 = \mathcal{G}_{-1,15}. & \end{array}$$

5. The equation $Q_n = \mathcal{G}_{m,g}$

If n is even, then from (2) we get

$$C_{Q_n}^{even}: \quad Y^2 = 2((g-2)X^2 - (g-4)X)^2 - 8,$$

and if n is odd, then

$$C_{Q_n}^{odd}: \quad Y^2 = 2((g-2)X^2 - (g-4)X)^2 + 8.$$

Here we have that $(1,0) \in C_{Q_n}^{even}(\mathbb{Q})$ and $(1,4) \in C_{Q_n}^{odd}(\mathbb{Q})$. Substituting $X = U + 1$ in case of $C_{Q_n}^{even}$ we get

$$Y^2 = U\left((2g^2 - 8g + 8)U^3 + (4g^2 - 8g)U^2 + (2g^2 + 8g - 16)U + 8g\right).$$

Hence, we have that

$$\begin{aligned}&U \mid 8g \text{ or}\\&(2g^2-8g+8)U^3+(4g^2-8g)U^2+(2g^2+8g-16)U+8g=\square \text{ or}\\&(2g^2-8g+8)U^3+(4g^2-8g)U^2+(2g^2+8g-16)U+8g=-\square.\end{aligned}$$

We have the following result.

Lemma 5.1. *The solutions of the equation* $Q_n=\mathcal{G}_{m,g}$ *with* $m,n\geq 0$ *and* $g\in\{6,8,9,10,\dots,20\}$ *are*

$$\begin{array}{lll}Q_0=\mathcal{G}_{1,g}, & Q_1=\mathcal{G}_{1,g}, \textit{where}\ g\in\{6,8,9,10,\dots,20\}, & \\ Q_2=\mathcal{G}_{-1,6}, & Q_3=\mathcal{G}_{-1,10}, & Q_4=\mathcal{G}_{2,17},\\ Q_{13}=\mathcal{G}_{-79,17}, & Q_6=\mathcal{G}_{-3,19}, & Q_4=\mathcal{G}_{-1,20}.\end{array}$$

Remark. *Siegel* [21] *in 1926 proved that the hyperelliptic equation*

$$y^2=a_0x^n+a_1x^{n-1}+\dots+a_n=:f(x)$$

has only a finite number of integer solutions if f has at least three simple roots. This result is ineffective, that is, its proof does not provide any algorithm for finding the solutions. By using explicit lower bounds for linear forms in logarithms Baker [2] *gave an effective version of the above result of Siegel. To show that for fixed g the equation*

$$\mathcal{G}_{m,g}=R_n, \quad R_n\in\{F_n,L_n,P_n,Q_n\}$$

has only finitely many effectively computable solutions, one has to prove that the quartic polynomials defining the genus 1 curves $C_{R_n}^{even}, C_{R_n}^{odd}$ have at least three simple roots. It is easy to see that the discriminant of these quartic polynomials can be zero only if $g=0$ or $g=2$, hence there are only finitely many effectively computable solutions for fixed $g>2$.

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Mathematical Institute, University of Debrecen and the Number Theory Research Group of the Hungarian Academy of Sciences, P.O. Box 12, 4010 Debrecen, Hungary
E-mail address: tengely@math.klte.hu