RELATIVELY PRIME PARTITIONS WITH TWO AND THREE PARTS

MOHAMED EL BACHRAOUI

Abstract. A set A of positive integers is relatively prime if $\gcd(A) = 1$. A partition of n is *relatively prime* if its parts form a relatively prime set. The number of partitions of n into exactly k parts is denoted by $p(n, k)$ and the number of relatively prime partitions into exactly k parts is denoted by $p_\Psi(n, k)$. In this note we give explicit formulas for $p_\Psi(n, 2)$ and $p_\Psi(n, 3)$ in terms of the prime divisors of n.

1. Introduction

In 1964, Gould [5] investigated compositions of positive integers whose parts are relatively prime. In 1990, Schmutz [7] considered the number a_n of partitions of a positive integer whose parts are pairwise relatively prime and found asymptotic estimates for $\log a_n$. In 2000, Nathanson [6] studied partitions with parts belonging to a nonempty finite set whose elements are relatively prime. Recently, Andrews [2] considered k-compositions with up to k copies of each part and the parts form a relatively prime set. Our main purpose in this work is to count the number of partitions of a positive integer into exactly three relatively prime parts.

Throughout, k and n will denote positive integers such that $k \leq n$ and p will range over prime numbers. We will use $\lfloor x \rfloor$ to denote the floor of x and $\langle x \rangle$ to denote the integer closest to x. A set A of positive integers is relatively prime if $\gcd(A) = 1$, and accordingly a partition of n is *relatively prime* if its parts form a relatively prime set. The number of partitions of n into exactly k parts is denoted by $p(n, k)$ and the number of relatively prime partitions into exactly k parts is denoted by $p_\Psi(n, k)$. Formulas for $p(n, k)$ for some small value of k can be found in [3]. It is immediately seen that $p(q, 3) = p_\Psi(q, 3)$ whenever q is a prime number. The following result is quite clear.

Theorem 1.1. *If $n > 2$, then*

$$p_\Psi(n, 2) = \frac{1}{2}\phi(n),$$

where ϕ is the Euler totient function,

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proof. The following identities are straightforward:

$$\begin{aligned} p_\Psi(n, 2) &= \#\{(a, b) : \ a < b, \gcd(a, b) = 1, \text{ and } a + b = n\} \\ &= \#\{a : \ a < n - a \text{ and } \gcd(a, n) = 1\} \\ &= \frac{1}{2}\#\{a : \ a < n \text{ and } \gcd(a, n) = 1\} \\ &= \frac{1}{2}\phi(n). \end{aligned}$$

This completes the proof. □

Corollary 1.2. *If n and m are relatively prime positive integers such that $nm > 2$, then*

$$p_\Psi(nm, 2) = 2p_\Psi(n, 2)p_\Psi(m, 2).$$

However, formulas for $p_\Psi(n, k)$ for larger values of k are far from straightforward. In this paper we derive an identity for $p_\Psi(n, 3)$ in terms of the prime factors of n. It is clear that

$$p(n, k) = \sum_{d|n} p_\Psi\left(\frac{n}{d}, k\right),$$

which by Möbius inversion is equivalent to

$$p_\Psi(n, k) = \sum_{d|n} \mu(d)p\left(\frac{n}{d}, k\right), \tag{1.1}$$

where $\mu(d)$ is the Euler μ function. Furthermore, it is well-known that the generating function for $p(n, k)$ is

$$\sum_{n\geq k} p(n, k)q^n = \frac{q^k}{(1-q)(1-q^2)\dots(1-q^k)}, \tag{1.2}$$

see for instance [1]. We shall combine (1.1) and (1.2) to get the formula for $p_\Psi(n, 3)$. We note that a combination of (1.1) and (1.2) can also be used to give an alternate proof for Theorem 1.1 on $p_\Psi(n, 2)$. This approach does not easily extend to $p_\Psi(n, k)$ for higher k since the formulas for $p(n, k)$ obtained from the generating function (1.2) are not so neatly seen as functions of n, see [3]. We recall the following formula on so-called Jordan totient function of order 2 which can be found for instance in [4],

$$J_2(n) := \sum_{d|n} \mu(d)\frac{n^2}{d^2} = n^2 \prod_{p|n}\left(1 - \frac{1}{p^2}\right). \tag{1.3}$$

2. Main Result

For $k = 3$, the generating function (1.2) yields,

$$p(n, 3) = \frac{n^2}{12} - \frac{7}{72} - \frac{1}{8}(-1)^n + \frac{2}{9}\cos\frac{2n\pi}{3} = \left\langle \frac{n^2}{12} \right\rangle. \tag{2.1}$$

Further it is easy to check that,

$$\left\langle \frac{n^2}{12} \right\rangle = \begin{cases} \frac{n^2}{12} & \text{if } n \equiv 0 \bmod 6, \\ \frac{n^2-1}{12} & \text{if } n \equiv 1 \text{ or } n \equiv 5 \bmod 6, \\ \frac{n^2-4}{12} & \text{if } n \equiv 2 \text{ or } n \equiv 4 \bmod 6, \\ \frac{n^2+3}{12} & \text{if } n \equiv 3 \bmod 6. \end{cases} \tag{2.2}$$

Throughout, we will be using (2.2) and the basic facts that the function μ is multiplicative, that $\mu(n) = 0$ whenever n has a nontrivial square factor, and that $\sum_{d|n} \mu(d) = 0$ whenever $n > 1$. We need the following lemma.

Lemma 2.1. *If $n \geq 4$, then*

$$\sum_{d|n} \mu(d)\left\langle \frac{n^2}{12d^2} \right\rangle = \frac{1}{12}\sum_{d|n} \mu(d)\left(\frac{n}{d}\right)^2.$$

Proof. We proceed by induction on n. For $n = 4$ the statement is trivial. Suppose now that the identity holds for all $4 \le k < n$. We only show the result for the cases $n \equiv 1 \bmod 6$, $n \equiv 5 \bmod 6$, and $n \equiv 3 \bmod 6$ and note that the remaining cases follow by similar arguments.

Case 1. Assume that $n \equiv 1 \bmod 6$ or $n \equiv 5 \bmod 6$. Then $\frac{n}{d} \equiv 1 \bmod 6$ or $\frac{n}{d} \equiv 5 \bmod 6$ for all $d|n$, and so by identity (2.2),

$$\begin{aligned}\sum_{d|n} \mu(d) \left\langle \frac{n^2}{12d^2} \right\rangle &= \sum_{d|n} \mu(d) \frac{\frac{n^2}{d^2} - 1}{12} \\ &= \frac{1}{12} \sum_{d|n} \mu(d) \left(\frac{n^2}{d^2}\right) - \frac{1}{12} \sum_{d|n} \mu(d) \\ &= \frac{1}{12} \sum_{d|n} \mu(d) \left(\frac{n^2}{d^2}\right).\end{aligned}$$

Case 2. Suppose that $n \equiv 3 \bmod 6$, say $n = 3^l m$ with $2 \nmid m$, $3 \nmid m$, and $l \ge 1$. If $l > 1$, then as $\frac{3^{l-1}m}{d} \equiv \frac{3^l m}{d} \equiv 3 \bmod 6$ for all $d|m$, we have by (2.2),

$$\left\langle \frac{(3^{l-1}m)^2}{12d^2} \right\rangle = \frac{\frac{(3^{l-1}m)^2}{d^2} + 3}{12} \text{ and } \left\langle \frac{(3^l m)^2}{12d^2} \right\rangle = \frac{\frac{(3^l m)^2}{d^2} + 3}{12}.$$

Then

$$\begin{aligned}\sum_{d|3^l m} \mu(d) \left\langle \frac{(3^l m)^2}{12d^2} \right\rangle &= \sum_{d|m} \mu(3d) \left\langle \frac{(3^l m)^2}{12(3d)^2} \right\rangle + \sum_{d|m} \mu(d) \left\langle \frac{(3^l m)^2}{12d^2} \right\rangle \\ &= -\sum_{d|m} \mu(d) \left\langle \frac{(3^{l-1}m)^2}{12d^2} \right\rangle + \sum_{d|m} \mu(d) \left\langle \frac{(3^l m)^2}{12d^2} \right\rangle \\ &= -\sum_{d|m} \mu(d) \frac{\frac{(3^{l-1}m)^2}{d^2} + 3}{12} + \sum_{d|m} \mu(d) \frac{\frac{(3^l m)^2}{d^2} + 3}{12} \\ &= -\frac{1}{12} \left(\sum_{d|m} \mu(d) \frac{(3^{l-1}m)^2}{d^2} + \sum_{d|m} \mu(d) \frac{(3^l m)^2}{d^2} \right) \\ &= \frac{1}{12} \left(\sum_{d|m} \mu(3d) \frac{(3^l m)^2}{(3d)^2} + \sum_{d|m} \mu(d) \frac{(3^l m)^2}{d^2} \right) \\ &= \frac{1}{12} \sum_{d|3^l m} \mu(d) \frac{(3^l m)^2}{d^2}.\end{aligned}$$

If $l = 1$, then since $\frac{3m}{d} \equiv 3 \bmod 6$ for all $d|m$, we find by (2.2),

$$\left\langle \frac{(3m)^2}{12d^2} \right\rangle = \frac{\frac{(3m)^2}{d^2} + 3}{12}.$$

Note that in this case $m > 1$ and so $\sum_{d|m} \mu(d) = 0$. Moreover, the induction hypothesis yields

$$\sum_{d|m} \mu(d) \left\langle \frac{m^2}{12d^2} \right\rangle = \frac{1}{12} \sum_{d|m} \mu(d) \frac{m^2}{d^2}.$$

Then

$$\begin{aligned}
\sum_{d|3m} \mu(d) \left\langle \frac{(3m)^2}{12d^2} \right\rangle &= \sum_{d|m} \mu(3d) \left\langle \frac{(3m)^2}{12(3d)^2} \right\rangle + \sum_{d|m} \mu(d) \left\langle \frac{(3m)^2}{12d^2} \right\rangle \\
&= -\sum_{d|m} \mu(d) \left\langle \frac{m^2}{12d^2} \right\rangle + \sum_{d|m} \mu(d) \left\langle \frac{(3m)^2}{12d^2} \right\rangle \\
&= -\frac{1}{12} \sum_{d|m} \mu(d) \frac{m^2}{d^2} + \sum_{d|m} \mu(d) \frac{\frac{(3m)^2}{d^2} + 3}{12} \\
&= \frac{1}{12} \sum_{d|m} \mu(3d) \frac{(3m)^2}{(3d)^2} + \frac{1}{12} \sum_{d|m} \mu(d) \frac{(3m)^2}{d^2} \\
&= \frac{1}{12} \sum_{d|3m} \mu(d) \frac{(3m)^2}{d^2}.
\end{aligned}$$

□

Theorem 2.2. *If $n \geq 4$, then*

$$p_\Psi(n, 3) = \frac{n^2}{12} \prod_{p|n} \left(1 - \frac{1}{p^2}\right).$$

Proof. By identities (1.1), (2.1), Lemma 2.1, and identity (1.3) we obtain

$$\begin{aligned}
p_\Psi(n, 3) &= \sum_{d|n} \mu(d) p\left(\frac{n}{d}, 3\right) \\
&= \sum_{d|n} \mu(d) \left\langle \frac{n^2}{12d^2} \right\rangle \\
&= \frac{1}{12} \sum_{d|n} \mu(d) \left(\frac{n}{d}\right)^2 \\
&= \frac{n^2}{12} \prod_{p|n} \left(1 - \frac{1}{p^2}\right).
\end{aligned}$$

This proves the theorem. □

Corollary 2.3. *If $n > 4$, then $p_\Psi(n, 3)$ is even.*

Corollary 2.4. *If n and m are relatively prime positive integers such that $nm \geq 4$, then*

$$p_\Psi(nm, 3) = 12 p_\Psi(n, 3) p_\Psi(m, 3).$$

Corollary 2.5. *If $n \geq 4$, then*

$$\frac{p_\Psi(n,3)}{p_\Psi(n,2)} = \frac{n}{6}\prod_{p|n}\left(1+\frac{1}{p}\right).$$

References

[1] G. E. Andrews, *The Theory of Partitions*, Cambridge University Press, 1998.
[2] G. E. Andrews, *The Theory of Compositions, IV: Multicompositions*, The Mathematics Student (Special Centenary Volume), (2007), 25–31.
[3] G. E. Andrews and K. Eriksson, *Integer Partitions*, Cambridge University Press, 2004.
[4] T. M. Apostol, *Introduction to Analytic Number Theory*, Undergraduate Texts in Mathematics, Springer, 1st edition, 1976.
[5] H. W. Gould, *Binomial Coefficients, The Bracket Function, and Compositions with Relatively Prime Summands*, The Fibonacci Quarterly, **2.3** (1964), 241–260.
[6] M. B. Nathanson, *Partitions with Parts in a Finite Set*, Proceedings of the American Mathematical Society, **128.5** (2000), 1269–1273.
[7] E. Schmutz, *Partitions Whose Parts are Pairwise Relatively Prime*, Discrete Mathematics, **81.1** (1990), 87–98.

MSC2000: 11A25, 05A17, 11P83

Department of Mathematical Sciences, United Arab Emirates University, PO Box 17551, Al-Ain, UAE
E-mail address: melbachraoui@uaeu.ac.ae