POWER SUM IDENTITIES WITH GENERALIZED STIRLING NUMBERS

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Abstract. We prove several combinatorial identities involving Stirling functions of the second kind with a complex variable. The identities also involve Stirling numbers of the first kind, binomial coefficients and harmonic numbers.

1. Introduction

Butzer, Kilbas and Trujillo [2] defined the Stirling functions of the second kind by

$$S(\alpha, k) = \frac{1}{k!}\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}j^{\alpha}, \tag{1.1}$$

for all complex numbers $\alpha \neq 0$ and all positive integers k. This definition is consistent with the definition given by Flajolet and Prodinger [5]. When $\alpha = n$ is a positive integer, $S(n, k)$ are the classical Stirling numbers of the second kind [3]. The purpose of this note is to prove the five power sum identities (2.3), (2.14), (2.17), (2.20) and (2.21) below involving the Stirling functions $S(\alpha, k)$. In fact, we describe a general method for obtaining such identities.

Recall that the *binomial transform* of a sequence $a_1, a_2, \dots$ is a new sequence $b_1, b_2, \dots,$ such that for every positive integer k,

$$b_k = \sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}a_j, \quad \text{with inversion} \quad a_k = \sum_{j=1}^{k}\binom{k}{j}b_j \tag{1.2}$$

[8, (5.48), p. 192], [9, 10]. In equation (1.2), we tacitly assume that $a_0 = b_0 = 0$. Equation (1.1) shows that the sequences $k!S(\alpha, k)$ and k^{α} are related by the binomial transform. The inversion formula then yields

$$k^{\alpha} = \sum_{j=1}^{k}\binom{k}{j}j!S(\alpha, j), \tag{1.3}$$

for any positive integer k.

2. The Identities

We start with a simple lemma.

Lemma 2.1. *Let $c_1, c_2, \dots,$ be a sequence of complex numbers. Then for every positive integer m we have*

$$\sum_{k=1}^{m}k^{\alpha}c_k = \sum_{j=1}^{m}j!S(\alpha, j)\sum_{k=j}^{m}\binom{k}{j}c_k. \tag{2.1}$$

Proof. For the proof we just need to use (1.3) for k^α and then change the order of summation on the right hand side

$$\sum_{k=1}^{m} k^\alpha c_k = \sum_{k=1}^{m} c_k \sum_{j=1}^{k} \binom{k}{j} j! S(\alpha, j) = \sum_{j=1}^{m} j! S(\alpha, j) \sum_{k=j}^{m} \binom{k}{j} c_k. \tag{2.2}$$

□

This lemma helps to generate power sum identities by using various upper summation identities. We present here five examples arranged in four propositions.

Proposition 2.2. *For every positive integer m and every two complex numbers $\alpha \neq 0$, x,*

$$\sum_{k=1}^{m} k^\alpha x^k = \sum_{j=1}^{m} j! S(\alpha, j) \sigma(x, m, j), \tag{2.3}$$

where $\sigma(x, m, j)$ is the (upper summation) polynomial

$$\sigma(x, m, j) = \sum_{k=j}^{m} \binom{k}{j} x^k = x^j \sum_{r=0}^{m-j} \binom{r+j}{j} x^r. \tag{2.4}$$

In particular, when $x = 1$ one has

$$\sum_{k=1}^{m} k^\alpha = \sum_{j=1}^{m} \binom{m+1}{j+1} j! S(\alpha, j). \tag{2.5}$$

Proof. We use the lemma with $c_k = x^k$. When $x = 1$ we use the upper summation identity

$$\sum_{k=j}^{m} \binom{k}{j} = \binom{m+1}{j+1} \tag{2.6}$$

(see, for instance, [7, 1.52] or [8, p. 174]). Thus (2.3) turns into (2.5). □

Remark 2.3. *Identity* (2.5) *was proved in* [2] *in the equivalent form*

$$\sum_{k=1}^{m} k^\alpha = \sum_{j=1}^{m} \binom{m}{j} (j-1)! S(\alpha+1, j) \tag{2.7}$$

by induction. The equivalence follows from the properties

$$S(\alpha+1, k) = kS(\alpha, k) + S(\alpha, k-1) \tag{2.8}$$

(*see* [2, 1.16]), *and the well-known binomial identity* [8, p. 174],

$$\binom{m}{k} + \binom{m}{k-1} = \binom{m+1}{k}. \tag{2.9}$$

Remark 2.4. *With complex powers $\alpha \neq 0$ as in* (2.3) *we have the flexibility to write*

$$\sum_{k=1}^{m} \frac{x^k}{k^\alpha} = \sum_{j=1}^{m} j! S(-\alpha, j) \sigma(x, m, j). \tag{2.10}$$

When $\alpha = n$ is a positive integer, identity (2.5) (*or* (2.7), *to that matter*) *is well-known and has a long history. In the early 18th century, Bernoulli evaluated $\sum_{k=1}^{m} k^n$ in terms of the numbers known today as Bernoulli numbers. Continuing Bernoulli's work, Leonard*

Euler [4, paragraphs 173, 176] *evaluated sums of the form* $\sum_{k=1}^{m} k^n x^k$*, essentially by applying* n *times the operator* $x\frac{d}{dx}$ *to the identity*

$$\sum_{k=1}^{m} x^k = \frac{1}{1-x} - \frac{x^{m+1}}{1-x} \tag{2.11}$$

($x \neq 1$)*. This led him to the discovery of a special sequence of polynomials* $A_k(x)$ *called today Eulerian polynomials* [1, 3, 6]*. In terms of these polynomials one has*

$$\left(x\frac{d}{dx}\right)^n \frac{1}{1-x} = \frac{A_n(x)}{(1-x)^{n+1}}, \qquad n = 0, 1, \ldots, \tag{2.12}$$

and therefore, with some help from the Leibniz rule

$$\sum_{k=1}^{m} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}} - x^{m+1} \sum_{k=0}^{n} \binom{n}{k} \frac{(m+1)^{n-k} A_k(x)}{(1-x)^{k+1}}. \tag{2.13}$$

This identity, however, cannot be extended to complex powers $n \to \alpha \in \mathbb{C}$ *for obvious reasons.*

The next identity can be viewed as the binomial transform of the sequence $k^\alpha x^k$ extending equation (1.1).

Proposition 2.5. *For every positive integer* m *and every two complex numbers* $\alpha \neq 0$, x,

$$\sum_{k=1}^{m} \binom{m}{k} k^\alpha x^k = \sum_{j=1}^{m} \binom{m}{j} j! S(\alpha, j) x^j (1+x)^{m-j}. \tag{2.14}$$

Proof. We apply the lemma with $c_k = \binom{m}{k} x^k$. The result then follows from the interesting identity

$$\sum_{k=j}^{m} \binom{m}{k} \binom{k}{j} x^k = \binom{m}{j} x^j (1+x)^{m-j}, \tag{2.15}$$

which is listed as number 3.118 on p. 36 in [7]. To prove this identity one can start by reducing both sides by x^j and then expanding $(1+x)^{m-j}$. □

Note that when $x = -1$, (2.14) turns into (1.1).

Remark 2.6. *Identity* (2.14) *for positive integers* $\alpha = r$ *can also be found in the treasure chest* [7]*. It is listed there* (*as number 1.126 on p.16*) *in the form*

$$\sum_{k=0}^{n} \binom{n}{k} k^r x^k = (1+x)^n \sum_{j=0}^{r} (-1)^j \binom{n}{j} \frac{x^j}{(1+x)^j} \sum_{k=0}^{j} (-1)^k \binom{j}{k} k^r. \tag{2.16}$$

Note that in (2.16) *the number* r *has to be a positive integer, because it stands for the upper limit of the first sum on the RHS. For the case* $x = 1$, (2.16) *was recently rediscovered by Spivey* [10].

The next identity involves the unsigned Stirling numbers of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ [8].

Proposition 2.7. *For every positive integer* m *and every complex* $\alpha \neq 0$ *we have*

$$\sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix} k^\alpha = \sum_{j=1}^{m} j! S(\alpha, j) \begin{bmatrix} m+1 \\ j+1 \end{bmatrix}. \tag{2.17}$$

Proof. The proof uses the lemma with $c_k = \begin{bmatrix} m \\ k \end{bmatrix}$ and also the upper summation identity [8, (6.16), p. 265]

$$\sum_{k=j}^{m} \binom{k}{j} \begin{bmatrix} m \\ k \end{bmatrix} = \begin{bmatrix} m+1 \\ j+1 \end{bmatrix}. \tag{2.18}$$

□

We finish this note with two identities involving the harmonic numbers

$$H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \quad (k = 1, 2, \ldots). \tag{2.19}$$

Proposition 2.8. *For every positive integer m and every complex power $\alpha \neq 0$,*

$$\sum_{k=1}^{m} H_k k^\alpha = \sum_{j=1}^{m} j! S(\alpha, j) \binom{m+1}{j+1} \left(H_{m+1} - \frac{1}{j+1} \right), \tag{2.20}$$

$$\sum_{k=1}^{m} \frac{k^\alpha}{m-k+1} = \sum_{j=1}^{m} j! S(\alpha, j) \binom{m+1}{j} (H_{m+1} - H_j). \tag{2.21}$$

Proof. This follows from the lemma with $c_k = H_k$ and $c_k = \dfrac{1}{m-k+1}$ correspondingly and also from the two upper summation identities [8, (6.70), p. 280 and p. 354],

$$\sum_{k=j}^{m} \binom{k}{j} H_k = \binom{m+1}{j+1} \left(H_{m+1} - \frac{1}{j+1} \right) \tag{2.22}$$

$$\sum_{k=j}^{m} \binom{k}{j} \frac{1}{m-k+1} = \binom{m+1}{j} (H_{m+1} - H_j). \tag{2.23}$$

□

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References

[1] K. N. Boyadziev, *Apostol-Bernoulli Functions, Derivative Polynomials and Eulerian Polynomials*, Advances and Applications in Discrete Mathematics, **1.2** (2008), 109–122.

[2] P. L. Butzer, A. A. Kilbas and J. J. Trujillo, *Stirling Functions of the Second Kind in the Setting of Difference and Fractional Calculus*, Numerical Functional Analysis and Optimization, **24.7-8** (2003), 673–711.

[3] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, rev. enl. ed. Dordrecht, Netherlands: Reidel, pp. 137–139, 1974.

[4] L. Eulero, *Institutiones Calculi Differentialis cum ejus usu in Analysi Finitorum ac Doctrina Serierum*, Impensis Academiae Imperialis Scientiarum Petropolitanae, 1755. Also, another edition, Ticini: in Typographeo Petri Galeatii Superiorum Permissu, 1787. (Opera Omnis Ser. I (Opera Math.), Vol. X, Teubner, 1913). Online at `http://www.math.dartmouth.edu/~euler/pages/E212.html`.

[5] P. Flajolet and H. Prodinger, *On Stirling Numbers for Complex Arguments and Hankel Contours*, SIAM J. Discrete Math, **12.2** (1999), 155–159.

[6] D. Foata, *Les Polynômes Eulériens, d'Euler à Carlitz, Leonhard Euler, Mathématicien, Physicien et Théoricien de la Musique* (Conference proceedings, IRMA Strasbourg, November 15–16, 2007, ed. X. Hascher and A. Papadopoulos).

[7] H. W. Gould, *Combinatorial Identities*, Published by the author, Revised edition, 1972.
[8] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley Publ. Co., New York, 1994.
[9] D. E. Knuth, *The Art of Computer Programming Vol. 3*, Addison-Wesley, Reading, MA, (1973).
[10] M. Z. Spivey, *Combinatorial Sums and Finite Differences*, Discrete Math., **307** (2007), 3130–3146.

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