CONTINUED FRACTIONS OF ROOTS OF FIBONACCI-LIKE FRACTIONS

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Abstract. We describe the initial terms in the continued fraction expansion of numbers of the form $\sqrt[k]{\frac{a_{n+k}}{a_n}}$. Here, (a_n) is a sequence satisfying $a_{n+1} = ba_n + a_{n-1}$ for a positive integer b, and k is a term in the sequence $F_{b,n}$ satisfying the same recurrence relation, with $F_{b,0} = 0$ and $F_{b,1} = 1$. Our results generalize previous work of the second author concerning the initial terms in the continued fraction expansion of $\sqrt[5]{\frac{F_{n+5}}{F_n}}$.

1. Introduction

In [1], the second author looked at the continued fractions of numbers of the form $\sqrt[5]{\frac{F_{n+5}}{F_n}}$, where F_n denotes the nth Fibonacci number. Using Binet's formula, one can see that for any integer k,

$$\lim_{n\to\infty} \sqrt[k]{\frac{F_{n+k}}{F_n}} = \phi,$$

where ϕ is the golden ratio. Since the continued fraction expansion of ϕ consists of an endless sequence of 1's, the continued fraction of $\sqrt[k]{\frac{F_{n+k}}{F_n}}$ begins with a large number of 1's, when n is large. But when $k = 5$, something unusual happens. Using the compact notation

$$[a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots}}$$

for continued fractions, we list below the beginning terms of the continued fractions of $\sqrt[5]{\frac{F_{n+5}}{F_n}}$, for values of n ranging from 1 to 6.

$$\begin{gathered}
[1, 1, 1, 15, 2, 2, \ldots] \\
[1, 1, 2, 30, 2, 3, \ldots] \\
[1, 1, 1, 1, 1, 91, 2, 48, \ldots] \\
[1, 1, 1, 1, 2, 229, 2, 12, \ldots] \\
[1, 1, 1, 1, 1, 1, 1, 612, 1, 1, \ldots] \\
[1, 1, 1, 1, 1, 1, 2, 1593, 2, 18, \ldots].
\end{gathered}$$

When n is odd, we see a sequence of $n+2$ 1's followed by a large number; when n is even, we see a sequence of n 1's, followed by a 2, followed by a large number. In fact, the second author [1] proved the following theorem.

Theorem 1.1.

$$\sqrt[5]{\frac{F_{n+5}}{F_n}} = \begin{cases} [\overbrace{1,1,\cdots,1}^{n+2\ 1\text{'s}}, F_{2n+5}+2,\cdots] & n \text{ odd} \\ [\overbrace{1,1,\cdots,1}^{n\ 1\text{'s}}, 2, F_{2n+5}-4,\cdots] & n \text{ even.} \end{cases}$$

After seeing this theorem, people typically wonder what is so special about the number 5. Does one see anything interesting when one considers the continued fraction expansion of $\sqrt[k]{\frac{F_{n+k}}{F_n}}$ for larger values of k?

There are also other possible generalizations that one might imagine. It is natural to consider the continued fraction expansion of $\sqrt[k]{\frac{a_{n+k}}{a_n}}$, where the sequence (a_n) satisfies the Fibonacci recurrence relation $a_{n+1} = a_n + a_{n-1}$, but a_0 and a_1 are integers different from 0 and 1; for example, the Lucas numbers are generated in this way with $a_0 = 2$ and $a_1 = 1$.

Finally, suppose b is a positive integer, and let $F_{b,n}$ be the sequence defined by $F_{b,n+1} = bF_{b,n} + F_{b,n-1}$, where $F_{b,0} = 0$ and $F_{b,1} = 1$. Then, the continued fraction expansion of $F_{b,n+1}/F_{b,n}$ consists of n b's. One might expect that for certain interesting values of k, the continued fraction expansion of $\sqrt[k]{\frac{F_{b,n+k}}{F_{b,n}}}$ begins with a string of b's, followed by a large number.

Our paper is organized as follows. In Section 2, we list some examples, and describe our initial guesses about what these continued fractions look like in general. We hope that this section will give the reader a good sense of the questions involved in this paper. Our two main theorems, Theorems 3.1 and 3.2 are described in Section 3. These two theorems together cover all the cases that we consider in Section 2, and we show how they confirm our initial guesses. In Section 4, we sketch some of the background needed to prove our theorems, and we give proofs of each of these in Sections 5 and 6.

2. Explorations and guesses

The author of [1] made some observations (without proof) concerning the continued fraction expansions of $\sqrt[k]{\frac{F_{n+k}}{F_n}}$ for larger values of k.

(1) For $n \geq 15$, $\sqrt[13]{\frac{F_{n+13}}{F_n}} = [\overbrace{1,1,\ldots,1}^{n+2}, 2, 1, 377, \ldots]$.

(2) For $n \geq 19$, $\sqrt[34]{\frac{F_{n+34}}{F_n}} = [\overbrace{1,1,\ldots,1}^{n+4}, 12921, \ldots]$.

(3) For $n \geq 21$, $\sqrt[89]{\frac{F_{n+34}}{F_n}} = [\overbrace{1,1,\ldots,1}^{n+4}, 2, 1, 17710, \ldots]$.

(4) For $n \geq 27$, $\sqrt[233]{\frac{F_{n+233}}{F_n}} = [\overbrace{1,1,\ldots,1}^{n+6}, 606966, \ldots]$.

Note that in each case, $k = F_{2t+1}$ for some t. (We have listed the cases where t runs from 3 to 6.) Some readers might notice that 377 is a Fibonacci number and 17710 is one less than a Fibonacci number, though it is more difficult to see the relation between the numbers 12921 and 606966 and the Fibonacci numbers. This result contrasts with the result of [1] in that for a fixed value of k, letting n get larger, the large numbers in the continued fraction expansions eventually stabilize to a single value, rather than growing with n.

The examples seem to fall into two cases according to the parity of t; instead, one might look at the case in which $k = F_{4t-1}$ separately from the case where $k = F_{4t+1}$. In the case $k = F_{4t-1}$, $t \geq 2$, the first few large numbers one sees in the continued fraction expansions are

$$377, 17710, 832039, 39088168, 1836311902.$$

In the case $k = F_{4t+1}$, $t \geq 2$, the first few large numbers are

$$12921, 606966, 28514436, 1339571481, 62931345126.$$

The data suggests that for $k = F_{4t-1}$, $t \geq 2$, the continued fraction expansion of $\sqrt[k]{\frac{F_{n+k}}{F_n}}$

- begins with $n + 2t - 2$ 1's
- followed by $2, 1$
- followed by $F_{8t-2} - 1$ (unless $t = 2$, when we get $F_{8t-2} = 377$).

Likewise, for $k = F_{4t+1}$, $t \geq 2$, the corresponding continued fraction

- begins with $n + 2t$ 1's
- followed by $5 \cdot F_{8t+2} + 1$.

We next consider Lucas numbers, and we list below the continued fraction expansion of $\sqrt[5]{\frac{L_{n+5}}{L_n}}$ where n varies from 1 to 10, and L_n denotes the nth Lucas number.

$$[1, 1, 3, 1, \ldots]$$
$$[1, 1, 1, 2, 1, 6, \ldots]$$
$$[1, 1, 1, 1, 3, 16, \ldots]$$
$$[1, 1, 1, 1, 1, 2, 1, 46, \ldots]$$
$$[1, 1, 1, 1, 1, 1, 3, 121, \ldots]$$
$$[1, 1, 1, 1, 1, 1, 1, 2, 1, 319, \ldots]$$
$$[1, 1, 1, 1, 1, 1, 1, 1, 3, 835, \ldots]$$
$$[1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 2189, \ldots]$$
$$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 5730, \ldots]$$
$$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 15004, \ldots].$$

As in the case of fifth roots involving Fibonacci numbers, the large number one encounters grows with n. One sees similar behavior if one defines a sequence a_n using the Fibonacci recurrence relation, but letting a_0 and a_1 be another pair of integers. For example, if $a_0 = 5$ and $a_1 = 3$, the continued fraction of $\sqrt[5]{\frac{a_{25}}{a_{20}}}$ begins as follows:

$$[\overbrace{1, 1, \ldots, 1}^{21}, 2, 1, 1, 1, 43802030, \ldots].$$

After looking at a lot of data, we guessed that if

- a_0 and a_1 are relatively prime, and
- $a_0 > a_1 > 0$

then, in general, when n is sufficiently large, the continued fraction of $\sqrt[5]{\frac{a_{n+5}}{a_n}}$

- begins with $n + 1$ 1's
- followed by one of the two continued fractions for $\frac{a_2}{a_1}$

- followed by a number close to $\frac{a_{n+2}^2+a_{n+3}^2}{(a_0a_2-a_1^2)^2}$.

We also looked at the continued fraction expansions of numbers like $\sqrt[k]{\frac{L_{n+k}}{L_n}}$ for some larger values of k:

(1) For $n \geq 15$, $\sqrt[13]{\frac{L_{n+13}}{L_n}} = [\overbrace{1, 1, \ldots, 1}^{n+3}, 1892, \ldots]$.

(2) For $n \geq 18$, $\sqrt[34]{\frac{L_{n+34}}{L_n}} = [\overbrace{1, 1, \ldots, 1}^{n+3}, 2, 1, 2583, \ldots]$.

(3) For $n \geq 24$, $\sqrt[89]{\frac{L_{n+89}}{L_n}} = [\overbrace{1, 1, \ldots, 1}^{n+5}, 88556, \ldots]$.

(4) For $n \geq 26$, $\sqrt[233]{\frac{L_{n+233}}{L_n}} = [\overbrace{1, 1, \ldots, 1}^{n+5}, 2, 1, 121392, \ldots]$.

Again, we looked at the general case of a sequence a_n satisfying the recurrence relation $a_{n+1} = a_n + a_{n-1}$, and also satisfying the conditions mentioned earlier ($\gcd(a_0, a_1) = 1$ and $a_0 > a_1 > 0$). We found that when $k = F_{4t+1}$, $t \geq 2$, then for n sufficiently large, the continued fraction expansion of $\sqrt[k]{\frac{a_{n+k}}{a_n}}$

- begins with $n + 2t - 1$ 1's,
- followed by one of the two continued fractions of $\frac{a_2}{a_1}$
- followed by a number close to $\frac{5F_{8t+2}}{|a_0a_2-a_1^2|}$.

For $k = F_{4t-1}$, $t \geq 2$, the continued fraction of $\sqrt[k]{\frac{a_{n+k}}{a_n}}$

- begins with $n + 2t - 2$ 1's,
- followed by one of the continued fractions of $\frac{a_3+a_1}{a_2+a_0}$ (which may itself begin with a string of 1's)
- followed by a number close to $\frac{\gcd(a_3+a_1,a_2+a_0)^2 F_{8t+2}}{|a_0a_2-a_1^2|}$.

As another direction for generalizing Theorem 1.1, suppose b is a positive integer, and consider the sequence $F_{b,n}$ introduced in Section 1. As we mentioned there, the continued fraction expansion of $F_{b,n+1}/F_{b,n}$ consists of n b's. One might expect that for certain interesting values of k, the continued fraction expansion of $\sqrt[k]{\frac{F_{b,n+k}}{F_{b,n}}}$ begins with a string of b's, followed by a large number. We list some examples below where we suppose $b = 3$.

(1) For $n \geq 6$, $\sqrt[10]{\frac{F_{3,n+10}}{F_{3,n}}} = [\overbrace{3, 3, \ldots, 3}^{n+1}, 1, 1, 1, 359, \ldots]$.

(2) For $n \geq 11$, $\sqrt[109]{\frac{F_{3,n+109}}{F_{3,n}}} = [\overbrace{3, 3, \ldots, 3}^{n+2}, 556884, \ldots]$.

(3) For $n \geq 14$, $\sqrt[1189]{\frac{F_{3,n+1189}}{F_{3,n}}} = [\overbrace{3, 3, \ldots, 3}^{n+3}, 1, 1, 1, 5097242, \ldots]$.

(4) For $n \geq 19$, $\sqrt[12970]{\frac{F_{3,n+12970}}{F_{3,n}}} = [\overbrace{3, 3, \ldots, 3}^{n+4}, 7884878043, \ldots]$.

In this case, the numbers $10, 109, 1189, 12970$ are all of the form $F_{b,2t+1}$ for some t. Note that one might also look at a sequence a_n defined using the same recurrence relation, but with different starting values.

We refrain from describing our initial guesses in these cases, believing instead that the reader is ready to see some results.

3. Results

We wish to consider nontrivial sequences (a_n) of integers that satisfy the recurrence relation $a_{n+1} = ba_n + a_{n-1}$ for some positive integer b, but whose initial two terms are not specified. Note that multiplying or dividing all the terms of a given sequence by a single integer does not change the numbers $\sqrt[k]{\frac{a_{n+k}}{a_n}}$. Also, note that if $\gcd(a_0, a_1) \neq 1$, then because of the recurrence relation, this common divisor must also divide all the other terms in the sequence. These two comments imply that we can restrict our attention to sequences satisfying $\gcd(a_0, a_1) = 1$.

Also, it is easy to check that $\lim_{n\to\infty} a_n$ must be either ∞ or $-\infty$. Dividing all the terms in a sequence in the latter class by -1 yields a sequence in the former class. Once again, since such division does not change the numbers $\sqrt[k]{\frac{a_{n+k}}{a_n}}$, we can restrict our attention to sequences which tend to positive infinity. By shifting index, we can then assume that $a_0 \geq 0$ and $a_1 > 0$.

We call a sequence of integers (a_n) *Fibonacci-like* provided that

- there is a positive integer b so that $a_{n+1} = ba_n + a_{n-1}$ for all n,
- $\gcd(a_0, a_1) = 1$, and
- $a_0 \geq 0$ and $a_1 > 0$.

For example, both the classical Fibonacci sequence and the Lucas sequence are Fibonacci-like, with $b = 1$.

Recall now that any rational number greater than 1 has two distinct continued fraction expansions, one of odd length and one of even length. We will only use $[1]$ as the continued fraction expansion of 1 itself; that is, we will not consider $[0, 1]$.

Theorem 3.1. *Suppose (a_n) is a Fibonacci-like sequence, with b some positive integer, and suppose $k = F_{b,4t+1}$ for some positive integer t. Also, suppose that either b or t is greater than 1 (so $k > 5$). Then for n sufficiently large, the continued fraction expansion of $\sqrt[k]{\frac{a_{n+k}}{a_n}}$ takes the form*

$$[\overbrace{b, b, \cdots, b}^{n+2t-1\ b\text{'s}}, c_1, c_2, \ldots, c_l, c_{l+1}, \ldots]$$

where $[c_1, c_2, \ldots, c_l]$ is one of the two continued fraction expansions of $\frac{a_2}{a_1}$. In particular, if $a_0a_2 - a_1^2 > 0$, then l is even, while if $a_0a_2 - a_1^2 < 0$, then l is odd.

Finally, the distance between c_{l+1} and

$$\left\lfloor \frac{\phi_b^k \cdot k \cdot \sqrt{b^2+4}}{F_{b,k-4t-1}|a_0a_2 - a_1^2|} - 1 \right\rfloor$$

is at most 1.

If $k = F_{b,4t-1}$ for some positive integers b and t, then for n sufficiently large, the continued fraction expansion of $\sqrt[k]{\frac{a_{n+k}}{a_n}}$ takes the form

$$[\overbrace{b, b, \cdots, b}^{n+2t-3\ b\text{'s}}, c_1, c_2, \ldots, c_l, c_{l+1}, \ldots]$$

where $[c_1, c_2, \ldots, c_l]$ *is one of the two continued fraction expansions of* $\frac{a_4+a_2}{a_3+a_1}$. *In particular, if* $a_0a_2 - a_1^2 > 0$, *then* l *is even, while if* $a_0a_2 - a_1^2 < 0$, *then* l *is odd.*

Finally, the distance between c_{l+1} *and*

$$\left\lfloor \frac{\phi_b^k \cdot k \cdot \gcd(a_3 + a_1, a_2 + a_0)^2}{\sqrt{b^2+4} \cdot F_{b,k-4t+1}|a_0a_2 - a_1^2|} - 1 \right\rfloor$$

is at most 1.

Note that in the second case of the theorem above, we always have $\frac{a_4+a_2}{a_3+a_1} > 1$, so that this fraction always has two continued fraction expansions. In the first case, we only have $\frac{a_2}{a_1} \geq 1$. However, if $\frac{a_2}{a_1} = 1$, so that $\frac{a_2}{a_1}$ has only one continued fraction expansion of odd length, then since $\gcd(a_1, a_2) = 1$, we must have $a_1 = a_2 = 1$. Since $a_0 \geq 0$ and $a_2 = ba_1 + a_0$, we must also have $a_0 = 0$ and $b = 1$. Thus, $a_0a_2 - a_1^2$ must equal -1, and the requirement of the theorem that l be odd can be achieved.

We also observe that the continued fraction expansion of $\frac{a_2}{a_1}$ or $\frac{a_4+a_2}{a_3+a_1}$ may themselves begin with a b, but this is not always the case. For example, if $b = 1$, $a_1 = 1$, and $a_2 = 4$, then $\frac{a_2}{a_1} = 4$ and $\frac{a_4+a_2}{a_3+a_1} = \frac{13}{6}$.

Recall that in Section 2, we guessed the structure of the continued fraction for $\sqrt[k]{\frac{F_{n+k}}{F_n}}$, where $k = F_{4t+1}$ or $k = F_{4t-1}$. This is the case of Theorem 3.1 where $b = 1$, $a_1 = 1$ and $a_2 = 1$. We will just consider the case $k = F_{4t+1}$; the other one is similar. In this case, $\frac{a_2}{a_1} = 1$, so that according to the theorem, the continued fraction should begin with $(n + 2t - 1) + 1 = n + 2t$ 1's, as we anticipated in our initial guess. Now, the large number appearing in Theorem 3.1 simplifies to

$$\left\lfloor \frac{\phi^k \cdot k \cdot \sqrt{5}}{F_{k-4t-1}} - 1 \right\rfloor.$$

We had expected to see a number close to $5 \cdot F_{8t+2} + 1$. Using Binet's formula, we observe that

$$\begin{aligned} 5F_{8t+2} + 1 &\approx \sqrt{5}\phi^{8t+2} + 1 = \sqrt{5}(\sqrt{5}\phi^{4t+1}k - 1) + 1 \\ &= \sqrt{5}\left(\frac{\phi^k \cdot k}{\phi^{k-4t-1}/\sqrt{5}} - 1\right) + 1 \approx \sqrt{5}\left(\frac{\phi^k \cdot k}{F_{k-4t-1}} - 1\right) + 1 \\ &= \left(\frac{\phi^k \cdot k \cdot \sqrt{5}}{F_{k-4t-1}} - 1\right) - (\sqrt{5} - 2). \end{aligned}$$

Here, we have used the symbol $\approx$ to indicate that we have made an approximation, dropping a large power of $\bar{\phi}$. Thus, the difference between the term we anticipated, which was $5F_{8t+2}+1$, and the term given by the theorem, which is $\left(\frac{\phi^k \cdot k \cdot \sqrt{5}}{F_{k-4t-1}} - 1\right)$, is approximately $\sqrt{5}-2$, or about 0.236.

In Section 2, we also looked at the structure of the continued fraction for $\sqrt[k]{\frac{a_{n+k}}{a_n}}$, where (a_n) is a sequence satisfying $a_{n+1} = a_n + a_{n-1}$, and $a_0 > a_1 > 0$, and $k = F_{4t+1}$ or $k = F_{4t-1}$. This case is covered by Theorem 3.1 where $b = 1$. We again just consider the case $k = F_{4t+1}$. Then, the initial terms in the continued fraction expansion predicted by the theorem agree with our guess. Moreover, the large number appearing in the theorem now simplifies to

$$\left\lfloor \frac{\phi^k \cdot k \cdot \sqrt{5}}{F_{k-4t-1}|a_0 a_2 - a_1^2|} - 1 \right\rfloor.$$

We had expected to see a number close to $\frac{5F_{8t+2}}{|a_0a_2-a_1^2|}$. Once again, using the same techniques as above, we obtain

$$\frac{5F_{8t+2}}{|a_0 a_2 - a_1^2|} \approx \left(\frac{\phi^k \cdot k \cdot \sqrt{5}}{F_{k-4t-1}|a_0 a_2 - a_1^2|} - 1 \right) + \left(1 - \frac{\sqrt{5}}{|a_0 a_2 - a_1^2|} \right).$$

Thus, the difference between the term we anticipated and the term given by the theorem is approximately $1 - \frac{\sqrt{5}}{|a_0a_2-a_1^2|}$. Since $a_0 > a_1$, $|a_0a_2 - a_1^2|$ is at least 5, and this difference is less than 1.

The first case of Theorem 3.1 is not applicable if $b = t = 1$. Indeed, in this case, $k = F_{b,4t+1} = 5$ and $F_{b,k-4t-1} = 0$. Notice that the expression for c_{l+1} in Theorem 3.1 has the term $F_{b,k-4t-1}$ appearing in the denominator! We consider this situation in our next theorem.

Theorem 3.2. *Suppose (a_n) is a Fibonacci-like sequence where $b = 1$. For n sufficiently large, the continued fraction expansion of $\sqrt[5]{\frac{a_{n+5}}{a_n}}$ takes the form*

$$[\overbrace{1, 1, \cdots, 1}^{n \text{ 1's}}, c_1, c_2, \ldots, c_l, c_{l+1}, \ldots]$$

where $[c_1, c_2, \ldots, c_l] = \frac{a_3}{a_2}$ and the parity of l is opposite to that of n.

Finally, the distance between c_{l+1} and

$$\left\lfloor \frac{a_{n+2}^2 + a_{n+3}^2}{(a_0 a_2 - a_1^2)^2} + (-1)^n \frac{7/\sqrt{5}}{a_0 a_2 - a_1^2} - 1 \right\rfloor$$

is at most 1.

Note that the continued fraction expansion of $\frac{a_3}{a_2}$ will typically begin with a 1 itself, but this will not be the case for the ordinary Fibonacci numbers. Then $\frac{a_3}{a_2} = 2$, so that if n is even, the continued fraction expansion for $\frac{a_3}{a_2}$ needed for the statement of the theorem must consist of a single 2.

In Section 2, we looked at the structure of the continued fraction for $\sqrt[5]{\frac{a_{n+5}}{a_n}}$, where (a_n) is a sequence satisfying $a_{n+1} = a_n + a_{n-1}$, but $a_0 > a_1$, so that (a_n) is not the Fibonacci sequence. This case is covered by Theorem 3.2. In this case, the continued fraction expansion of $\frac{a_3}{a_2}$ begins with a 1, followed by the continued fraction expansion of $\frac{a_2}{a_1}$. Thus, the Theorem predicts $n+1$ 1's, followed by the continued fraction expansion of $\frac{a_2}{a_1}$ (as we saw in our initial guess). Moreover, the expression $(-1)^n \frac{7/\sqrt{5}}{a_0a_2-a_1^2} - 1$ is less than 5, and substantially smaller when the sequence (a_n) is not just a shift of the Fibonacci sequence. Thus, the theorem confirms our guess in Section 2.

Remark 3.3. In Theorem 3.1, in the case $k = F_{b,4t+1}$, we wrote the continued fraction expansion of $\sqrt[k]{\frac{a_{n+k}}{a_n}}$ in the form

$$[\overbrace{b, b, \cdots, b}^{n+2t-1 \text{ b's}}, c_1, c_2, \ldots, c_l, c_{l+1}, \ldots]$$

where $[c_1, c_2, \ldots, c_l]$ is one of the two continued fraction expansions of $\frac{a_2}{a_1}$. We could instead write this continued fraction expansion in the form

$$[c_1, c_2, \ldots, c_l, c_{l+1}, \ldots]$$

where $[c_1, c_2, \ldots, c_l]$ is one of the two continued fraction expansions of a_{n+2t+1}/a_{n+2t}. In this case, if $a_0a_2 - a_1^2 > 0$, then we would need $n + 2t - 1 + l$ to be even (i.e. $n + l$ to be odd), while if $a_0a_2 - a_1^2 < 0$, then we would need $n + 2t - 1 + l$ to be odd (i.e. $n + l$ to be even).

Similarly, in the case $k = F_{b,4t-1}$, we could write the continued fraction expansion of $\sqrt[k]{\frac{a_{n+k}}{a_n}}$ in the form

$$[c_1, c_2, \ldots, c_l, c_{l+1}, \ldots]$$

where $[c_1, c_2, \ldots, c_l]$ is one of the continued fraction expansions of $\frac{a_{n+2t+1}+a_{n+2t-1}}{a_{n+2t}+a_{n+2t-2}}$. As above, if $a_0a_2 - a_1^2 > 0$, then we would need $n + l$ to be odd, while if $a_0a_2 - a_1^2 < 0$, then we would need $n + l$ to be even.

Finally, in Theorem 3.2, we could write the continued fraction expansion of $\sqrt[5]{\frac{a_{n+5}}{a_n}}$ in the form

$$[c_1, c_2, \ldots, c_l, c_{l+1}, \ldots]$$

where $[c_1, c_2, \ldots, c_l]$ is the continued fraction expansion of $\frac{a_{n+3}}{a_{n+2}}$ with l odd.

Remark 3.4. In fact, Theorem 3.1 holds for a given Fibonacci-like sequence (a_n) if and only if it holds for the shifted Fibonacci-like sequence (a'_n) given by $a'_n = a_{n+1}$. To see this, first observe that for any Fibonacci-like sequence, the expression $(-1)^n(a_na_{n+2} - a_{n+1}^2)$ is independent of n, since

$$\begin{aligned}
&(-1)^{n+1}(a_{n+1}a_{n+3} - a_{n+2}^2) \\
&= (-1)^{n+1}(a_{n+1}(ba_{n+2} + a_{n+1}) - a_{n+2}(ba_{n+1} + a_n)) \\
&= (-1)^{n+1}(a_{n+1}^2 - a_na_{n+2}) = (-1)^n(a_na_{n+2} - a_{n+1}^2).
\end{aligned}$$

Moreover, $\gcd(a_{n+5} + a_{n+3}, a_{n+4} + a_{n+2})$ does not depend on n. Indeed,

$$a_{n+5} + a_{n+3} = b(a_{n+4} + a_{n+2}) + a_{n+3} + a_{n+1},$$

so

$$\gcd(a_{n+5} + a_{n+3}, a_{n+4} + a_{n+2}) = \gcd(a_{n+4} + a_{n+2}, a_{n+3} + a_{n+1}).$$

In particular, it suffices to prove that the continued fraction expansion of $\sqrt[k]{\frac{a_{n+k}}{a_n}}$ takes the required form when n is odd and sufficiently large. Indeed, if n is even, then the result for $\sqrt[k]{\frac{a_{n+k}}{a_n}}$ follows from the corresponding result for $\sqrt[k]{\frac{a'_{n-1+k}}{a'_{n-1}}}$.

The same reasoning applies to Theorem 3.2, too.

Remark 3.5. Using Binet's formula, we obtain

$$\left\lfloor \frac{F_{n+2}^2+F_{n+3}^2}{(F_0F_2-F_1^2)^2} + (-1)^n \frac{7/\sqrt{5}}{F_0F_2-F_1^2} - 1 \right\rfloor = \begin{cases} F_{2n+5}+2 & n \text{ odd} \\ F_{2n+5}-5 & n \text{ even.} \end{cases}$$

Notice that $F_{2n+5}-5$ is within 1 of $F_{2n+5}-4$. Hence, in the case of Fibonacci numbers themselves, Theorem 3.2 follows from the main result in [1].

4. Preliminaries

Given a sequence of positive integers $a_1, a_2, \ldots, a_l$ and real number $\alpha > 1$, we will also use the notation $[a_1, a_2, \ldots, a_l, \alpha]$ to denote

$$a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_l + \frac{1}{\alpha}}}.$$

The actual continued fraction expansion of $[a_1, a_2, \ldots, a_l, \alpha]$ would then be the concatenation of the sequence $a_1, a_2, \ldots, a_l$ with the continued fraction of α.

The following lemma follows immediately from Theorems 1.3 and 1.4 in Olds' text [2] on continued fractions, or one could consult virtually any text on elementary number theory.

Lemma 4.1. *Suppose $c_1, c_2, \ldots, c_l$ are positive integers. For $1 \le i \le n$, let $\frac{p_i}{q_i}$ be equal to $[c_1, c_2, \ldots, c_i]$, written in lowest terms. Then for $2 \le i \le l$, $p_iq_{i-1} - p_{i-1}q_i = (-1)^i$ and for any positive real number α,*

$$[c_1, c_2, \ldots, c_l, \alpha] = \frac{p_l\alpha + p_{l-1}}{q_l\alpha + q_{l-1}}.$$

Corollary 4.2. *With the hypotheses and notations of Lemma 4.1, we have*

$$[c_1, c_2, \ldots, c_l, \alpha] = \frac{p_l}{q_l} + \frac{(-1)^{l+1}}{q_l^2\left(\alpha + \frac{q_{l-1}}{q_l}\right)} = \frac{p_lq_l\left(\alpha + \frac{q_{l-1}}{q_l}\right) + (-1)^{l+1}}{q_l^2\left(\alpha + \frac{q_{l-1}}{q_l}\right)}.$$

Proof. Using Lemma 4.1, we have

$$[c_1, c_2, \ldots, c_l, \alpha] - \frac{p_l}{q_l} = \frac{p_l\alpha + p_{l-1}}{q_l\alpha + q_{l-1}} - \frac{p_l}{q_l} = \frac{p_{l-1}q_l - p_lq_{l-1}}{q_l(q_l\alpha + q_{l-1})} = \frac{(-1)^{l+1}}{q_l^2(\alpha + \frac{q_{l-1}}{q_l})}.$$

□

Now suppose (a_n) satisfies the recurrence relation $a_{n+1} = ba_n + a_{n-1}$. Let $\phi_b = \frac{b+\sqrt{b^2+4}}{2}$, and let $\bar{\phi}_b = \frac{b-\sqrt{b^2+4}}{2} = -1/\phi_b$. These are the roots of the polynomial $x^2 - bx - 1$. We thus have the following analogue of Binet's formula.

Lemma 4.3.

$$a_n = \frac{(a_1 - \bar{\phi}_b a_0)\phi_b^n - (a_1 - \phi_b a_0)\bar{\phi}_b^n}{\sqrt{b^2+4}} = \frac{(a_1 + \phi_b^{-1} a_0)\phi_b^n - (-1)^n(a_1 - \phi_b a_0)\phi_b^{-n}}{\sqrt{b^2+4}}.$$

Proof. The sequence on the right side of the first equation satisfies both the initial conditions and the recurrence relation given for the sequence a_n. The second expression for a_n then follows since $\phi_b\bar{\phi}_b = -1$. □

5. Theorem 3.1

We will prove the equivalent version of Theorem 3.1 given in Remark 3.3. This version, while it hides the initial structure of the the continued fraction expansions, is easier to work with for the sake of proving the theorem.

We first outline the proof in the case when $k = F_{b,4t+1}$. We will only consider odd values for n, as we can do by Remark 3.4. If $a_0a_2 - a_1^2 > 0$, let $\sigma = 1$. If $a_0a_2 - a_1^2 < 0$, let $\sigma = -1$.

Now, for a given odd value of n, let $[c_1, c_2, \ldots, c_l]$ be the continued fraction expansion of $\frac{a_{n+2t+1}}{a_{n+2t}}$, where the parity of l is determined by $(-1)^l = \sigma$. (Thus, since n is odd, $l+n$ is odd if $a_0a_2 - a_1^2 > 0$ and $l + n$ is even if $a_0a_2 - a_1^2 < 0$.) Note that $c_1, c_2, \ldots, c_l$ all depend on n, but we suppress this dependence in our notation.

We will show that if

$$z = \left\lfloor \frac{\phi_b^k \cdot k \cdot \sqrt{b^2+4}}{F_{b,k-4t-1}|a_0a_2 - a_1^2|} - 1 \right\rfloor,$$

then for l even (i.e. $\sigma = 1$), and n sufficiently large, we have

$$[c_1, c_2, \ldots, c_l, z-1] < \sqrt[k]{\frac{a_{n+k}}{a_n}} < [c_1, c_2, \ldots, c_l, z+2] \tag{5.1}$$

while for l odd (i.e. $\sigma = -1$), and n sufficiently large, we have

$$[c_1, c_2, \ldots, c_l, z+2] < \sqrt[k]{\frac{a_{n+k}}{a_n}} < [c_1, c_2, \ldots, c_l, z-1]. \tag{5.2}$$

Observe that this implies that the continued fraction expansion of $\sqrt[k]{\frac{a_{n+k}}{a_n}}$ is of the form $[c_1, c_2, \ldots, c_l, c_{l+1}, \ldots]$, where c_{l+1} is equal to either $z-1$, z, or $z+1$, which is the statement of Theorem 3.1 (in the case $k = F_{b,4t+1}$).

Now, using the notation from Lemma 4.1 and the result of Corollary 4.2, inequality (5.1) reduces to

$$\frac{p_lq_l\left(z-1+\frac{q_{l-1}}{q_l}\right)-\sigma}{q_l^2\left(z-1+\frac{q_{l-1}}{q_l}\right)} < \sqrt[k]{\frac{a_{n+k}}{a_n}} < \frac{p_lq_l\left(z+2+\frac{q_{l-1}}{q_l}\right)-\sigma}{q_l^2\left(z+2+\frac{q_{l-1}}{q_l}\right)},$$

while inequality (5.2) reduces to the reverse inequalities. Equivalently, inequality (5.1) reduces to the pair of inequalities

$$a_n\left(p_lq_l\left(z-1+\frac{q_{l-1}}{q_l}\right)-\sigma\right)^k - a_{n+k}\left(q_l^2\left(z-1+\frac{q_{l-1}}{q_l}\right)\right)^k < 0, \tag{5.3}$$

and

$$0 < a_n\left(p_lq_l\left(z+2+\frac{q_{l-1}}{q_l}\right)-\sigma\right)^k - a_{n+k}\left(q_l^2\left(z+2+\frac{q_{l-1}}{q_l}\right)\right)^k \tag{5.4}$$

while inequality (5.2) reduces to the reverse inequalities.

Now let d be a positive parameter, and consider the expression

$$a_n(p_lq_ld - \sigma)^k - a_{n+k}(q_l^2d)^k. \tag{5.5}$$

Let $d_0 = \frac{\phi_b^k \cdot k \cdot \sqrt{b^2+4}}{F_{b,k-4t-1}|a_0a_2 - a_1^2|}$. We will show that for n sufficiently large and odd, expression (5.5) has the same sign as $\sigma(d - d_0)$. For example, if $\sigma = 1$, then for n sufficiently large and odd,

(5.5) is negative if $d < d_0$ and positive if $d > d_0$. Since $z = \lfloor d_0 - 1 \rfloor$ and $0 \le \frac{q_{l-1}}{q_l} \le 1$, we have
$$z - 1 + \frac{q_{l-1}}{q_l} < d_0 < z + 2 + \frac{q_{l-1}}{q_l}.$$
Thus, for n odd and sufficiently large, inequalities (5.3) and (5.4) (or the opposite inequalities if $\sigma = -1$) must hold, so that c_{l+1} must be either $z-1$, z, or $z+1$.

In order to show that d_0 has the needed property, we use Lemma 4.3 to consider (5.5) as a Laurent polynomial $f(x)$, where $x = \phi_b^n$. In particular, since n is odd,
$$a_n = \frac{(a_1 + \phi_b^{-1} a_0)x + (a_1 - \phi_b a_0)x^{-1}}{\sqrt{b^2+4}}.$$
Similarly,
$$a_{n+k} = \frac{(a_1 + \phi_b^{-1} a_0)\phi_b^k x + (-1)^k (a_1 - \phi_b a_0)\phi_b^{-k} x^{-1}}{\sqrt{b^2+4}}.$$
Also, $p_l = a_{n+2t+1}$ and $q_l = a_{n+2t}$, so we have
$$p_l = \frac{(a_1 + \phi_b^{-1} a_0)\phi_b^{2t+1} x - (a_1 - \phi_b a_0)\phi_b^{-2t-1} x^{-1}}{\sqrt{b^2+4}},$$
and
$$q_l = \frac{(a_1 + \phi_b^{-1} a_0)\phi_b^{2t} x + (a_1 - \phi_b a_0)\phi_b^{-2t} x^{-1}}{\sqrt{b^2+4}}.$$
Notice that the exponents of $f(x)$ are all odd and range between $-2k-1$ and $2k+1$. In fact, the coefficients of x^{2k+1} in $a_n(p_l q_l d - \sigma)^k$ and $a_{n+k}(q_l^2 d)^k$ are both equal to
$$\frac{(a_1 + \phi_b^{-1} a_0)^{2k+1}}{\sqrt{b^2+4}^{2k+1}} \phi_b^{(4t+1)k} d^k.$$
Thus, the coefficient of x^{2k+1} in $f(x)$ is 0.

We now consider the coefficient of x^{2k-1} in $f(x)$.

Lemma 5.1. *The coefficient of x^{2k-1} in $f(x)$ is equal to $\sigma\alpha(\beta d - \gamma)$, where*
$$\alpha = \frac{\phi_b^{(4tk-4t-1)}(a_1 + \phi_b^{-1} a_0)^{2k-1} d^{k-1}}{(b^2+4)^k},$$
$$\beta = F_{b,k-4t-1}|a_0 a_2 - a_1^2|,$$
$$\gamma = \phi_b^k k \sqrt{b^2+4}.$$

Proof. It is a straightforward computation to show that the coefficient of x^{2k-1} in $a_n(p_l q_l d - \sigma)$ is
$$\begin{aligned}&\frac{(a_1 - \phi_b a_0)}{\sqrt{b^2+4}} \cdot \frac{(a_1 + \phi_b^{-1} a_0)^{2k} \phi_b^{(4t+1)k}}{(b^2+4)^k} \cdot d^k \\ &\qquad + \frac{a_1 + \phi_b^{-1} a_0}{\sqrt{b^2+4}} \cdot k \cdot \frac{a_1 + \phi_b^{-1} a_0)^{2(k-1)} \phi_b^{(4t+1)(k-1)}}{(b^2+4)^{k-1}} \\ &\qquad \cdot \left(\frac{b(a_1 + \phi_b^{-1} a_0)(a_1 - \phi_b a_0) d}{b^2+4} - \sigma \right)\end{aligned} \tag{5.6}$$

while the coefficient of x^{2k-1} in $a_{n+k}(q_l^2 d)^k$ is

$$\frac{(-1)^k(a_1-\phi_b a_0)\phi_b^{-k}}{\sqrt{b^2+4}}\cdot\frac{(a_1+\phi_b^{-1}a_0)^{2k}\phi_b^{4tk}}{(b^2+4)^k}\cdot d^k + \frac{(a_1+\phi_b^{-1}a_0)\phi_b^k}{\sqrt{b^2+4}}\cdot k\cdot\left(\frac{(a_1+\phi_b^{-1}a_0)^{2(k-1)}\phi_b^{4t(k-1)}}{(b^2+4)^{k-1}}\right)\cdot d^{k-1} \cdot\left(\frac{2(a_1+\phi_b^{-1}a_0)(a_1-\phi_b a_0)}{b^2+4}\cdot d\right). \tag{5.7}$$

Factoring out $\sigma\alpha$ from the difference of the above two coefficients yields an expression of the form $\beta d-\gamma$ where $\gamma=\phi_b^k k\sqrt{b^2+4}$ and β can be written as

$$-\sigma(a_1-\phi_b a_0)(a_1+\phi_b^{-1}a_0)\cdot\frac{-\phi_b^{k+4t+1}-kb\phi_b^k+(-1)^k\phi_b^{-k+4t+1}+2k\phi_b^{k+1}}{\sqrt{b^2+4}}.$$

Finally, it is straightforward to show that

$$-\sigma(a_1-\phi_b a_0)(a_1+\phi_b^{-1}a_0)=|a_0a_2-a_1^2|,$$

and

$$\frac{-\phi_b^{k+4t+1}-kb\phi_b^k+(-1)^k\phi_b^{-k+4t+1}+2k\phi_b^{k+1}}{\sqrt{b^2+4}}=F_{b,k-4t-1}.$$

□

Assuming d is positive, it is clear that α is positive, since $a_0\ge 0$ and $a_1>0$. It is easy to show that $k=F_{b,4t+1}>4t+1$ provided b and t are not both equal to 1. Thus, $F_{b,k-4t-1}>0$, and therefore β is also positive.

Now recalling the definition of d_0, we see that $d_0=\gamma/\beta$. By Lemma 5.1, if $\sigma=1$, and $d>d_0$, then for x sufficiently large, we have $f(x)>0$. If $\sigma=1$ and $d<d_0$, then for x sufficiently large, we have $f(x)<0$. The opposite statements hold when $\sigma=-1$. In short, for x sufficiently large, $f(x)$ has the same sign as $\sigma(d-d_0)$. Recall that by construction of f,

$$f(\phi_b^n)=a_n(p_lq_ld-\sigma)^k-a_{n+k}(q_l^2d)^k.$$

Thus, for n sufficiently large, $a_n(p_lq_ld-\sigma)^k-a_{n+k}(q_l^2d)^k$ has the same sign as $\sigma(d-d_0)$.

As we explained earlier, this implies that for n odd and sufficiently large, inequalities (5.3) and (5.4) (or the opposite inequalities if $\sigma=-1$) must hold. In turn, this implies that inequality (5.1) holds if $\sigma=1$, while inequality (5.2) holds if $\sigma=-1$, which completes the proof of Theorem 3.1 in the case $k=F_{b,4t+1}$.

We now outline the proof in the case when $k=F_{b,4t-1}$. This proof is very similar to the proof above, so we will just describe the differences. In this case, for a given odd value of n, we let $[c_1,c_2,\dots,c_l]$ be the continued fraction expansion of $\frac{a_{n+2t+1}+a_{n+2t-1}}{a_{n+2t}+a_{n+2t-2}}$, where the parity of l is determined by $(-1)^l=\sigma$. Let $g=\gcd(a_3+a_1,a_2+a_0)$. By Remark 3.4,

$$\gcd(a_{n+2t+1}+a_{n+2t-1},a_{n+2t}+a_{n+2t-2})=g.$$

Then, in the notation of Lemma 4.1, we have

$$p_l=\frac{a_{n+2t+1}+a_{n+2t-1}}{g}\quad\text{and}\quad q_l=\frac{a_{n+2t}+a_{n+2t-2}}{g}.$$

Now as before, let d be a positive parameter. We will again consider the expression

$$a_n(p_l q_l d - \sigma)^k - a_{n+k}(q_l^2 d)^k. \tag{5.8}$$

If we multiply expression (5.8) by the positive integer g^{2k}, we get

$$a_n((p_l g)(q_l g)d - g^2\sigma)^k - a_{n+k}((q_l g)^2 d)^k. \tag{5.9}$$

We again use Lemma 4.3 to rewrite (5.9) as a Laurent polynomial $f(x)$, where $x = \phi_b^n$. Once again, the exponents of $f(x)$ are all odd and range between $-2k-1$ and $2k+1$. The coefficients of x^{2k+1} in $a_n((p_l g)(q_l g)d - g^2\sigma)^k$ and $a_{n+k}((q_l g)^2 d)^k$ are both equal to

$$\frac{(a_1 + \phi_b^{-1}a_0)^{2k+1}}{\sqrt{b^2+4}}\phi_b^{(4t-1)k}d^k.$$

Thus, the coefficient of x^{2k+1} in $f(x)$ is 0.

We now consider the coefficient of x^{2k-1} in $f(x)$. We omit the proof of the following lemma, as it is straightforward and analagous to that of Lemma 5.1.

Lemma 5.2. *The coefficient of x^{2k-1} in expression (5.9) is equal to $\sigma\alpha(\beta d - \gamma)$, where*

$$\alpha = \frac{\phi_b^{(4tk-4t-2k+1)}(a_1 + \phi_b^{-1}a_0)^{2k-1}d^{k-1}}{\sqrt{b^2+4}},$$
$$\beta = \sqrt{b^2+4}\cdot F_{b,k-4t+1}\cdot |a_0a_2 - a_1^2|,$$
$$\gamma = \phi_b^k \cdot k \cdot g^2.$$

As before, it is easy to see that α is positive. Also, $k = F_{b,4t-1} > 4t-1$ provided that b and t are not both 1; thus $F_{b,k-4t+1} > 0$. If $b = t = 1$, then $k - 4t + 1 = 2 - 4 + 1 = -1$, and $F_{1,-1} = 1$. Thus, β is positive. The remainder of the proof in this case is entirely analogous to the case $k = F_{b,4t+1}$ proved above.

6. Theorem 3.2

Our proof of Theorem 3.2 is along the same lines as that of Theorem 3.1. Again, we may assume that n is odd and a_0 and a_1 are both positive. For a given odd n, we let $[c_1, c_2, \ldots, c_l]$ be the continued fraction expansion of $\frac{a_{n+3}}{a_{n+2}}$, where l is required to be odd. We will show that if

$$z_n = \left\lfloor \frac{a_{n+2}^2 + a_{n+3}^2}{(a_0a_2 - a_1^2)^2} + (-1)^n\frac{7/\sqrt{5}}{a_0a_2 - a_1^2} - 1 \right\rfloor$$

then we have

$$[c_1, c_2, \ldots, c_l, z_n + 2] < \sqrt[5]{\frac{a_{n+5}}{a_n}} < [c_1, c_2, \ldots, c_l, z_n - 1]. \tag{6.1}$$

This implies that the continued fraction expansion of $\sqrt[5]{\frac{a_{n+5}}{a_n}}$ takes the form

$$[c_1, c_2, \ldots, c_l, c_{l+1}, \ldots],$$

where c_{l+1} is equal to $z_n - 1$, z_n, or $z_n + 1$, which is the statement of Theorem 3.2.

Now, by Corollary 4.2, inequality (6.1) reduces to

$$\frac{p_l q_l\left(z_n + 2 + \frac{q_{l-1}}{q_l}\right) + 1}{q_l^2\left(z_n + 2 + \frac{q_{l-1}}{q_l}\right)} < \sqrt[5]{\frac{a_{n+5}}{a_n}} < \frac{p_l q_l\left(z_n - 1 + \frac{q_{l-1}}{q_l}\right) + 1}{q_l^2\left(z_n - 1 + \frac{q_{l-1}}{q_l}\right)},$$

or equivalently, to the pair of inequalities

$$a_n\left(p_lq_l\left(z_n+2+\frac{q_{l-1}}{q_l}\right)+1\right)^5-a_{n+5}\left(q_l^2\left(z_n+2+\frac{q_{l-1}}{q_l}\right)\right)^5<0, \tag{6.2}$$

and

$$0<a_n\left(p_lq_l\left(z_n-1+\frac{q_{l-1}}{q_l}\right)+1\right)^5-a_{n+5}\left(q_l^2\left(z_n-1+\frac{q_{l-1}}{q_l}\right)\right)^5. \tag{6.3}$$

Now, let d be a positive parameter, and consider the expression

$$a_n\left(p_lq_l\left(\frac{a_{n+2}^2+a_{n+3}^2}{(a_0a_2-a_1^2)^2}+d\right)+1\right)^5-a_{n+5}\left(q_l^2\left(\frac{a_{n+2}^2+a_{n+3}^2}{(a_0a_2-a_1^2)^2}+d\right)\right)^5. \tag{6.4}$$

Let

$$d_0=\frac{7/\sqrt{5}}{a_0a_2-a_1^2}.$$

We will show that for n sufficiently large and odd, expression (6.4) has the same sign as d_0-d.

Letting $x=\phi^n$, expression (6.4) can be written as a Laurent polynomial in x with odd exponents ranging from -21 to 21. To simplify our computations, we use *Mathematica* to find some of these coefficients. The coefficients of x^{21}, x^{19} and x^{17} are all zero. Let c_{15} denote the coefficient of x^{15}. After simplifying, we obtain

$$\sqrt{5}^{11}c_{15}=\phi^{35}\frac{a_1+\phi^{-1}a_0}{(a_0\phi-a_1)^7}(-7-\sqrt{5}(a_0^2+a_0a_1-a_1^2)d). \tag{6.5}$$

Note that

$$(a_0\phi-a_1)(a_0\phi^{-1}+a_1)=a_0^2+a_0a_1-a_1^2,$$

so that $(a_0\phi-a_1)$ has the same sign as $a_0^2+a_0a_1-a_1^2$. Thus, c_{15} is a decreasing linear function of d. Setting $d=d_0$ yields $c_{15}=0$, so that c_{15} has the same sign as d_0-d. Since

$$z_n=\left\lfloor\frac{a_{n+2}^2+a_{n+3}^2}{(a_0a_2-a_1^2)^2}+(-1)^n\frac{7/\sqrt{5}}{a_0a_2-a_1^2}-1\right\rfloor$$

and $0\le\frac{q_{l-1}}{q_l}\le 1$, we have

$$z_n-1+\frac{q_{l-1}}{q_l}<\frac{a_{n+2}^2+a_{n+3}^2}{(a_0a_2-a_1^2)^2}+(-1)^n\frac{7/\sqrt{5}}{a_0a_2-a_1^2}<z_n+2+\frac{q_{l-1}}{q_l}.$$

So, if n is odd and sufficiently large, inequalities (6.2) and (6.3) must hold, and hence c_{l+1} must be either z_n-1, z_n, or z_n+1. This completes the proof of Theorem 3.2.

References

[1] C. P. French, *Fifth Roots of Fibonacci Fractions*, The Fibonacci Quarterly, **44.3** (2006), 209–215.
[2] C. D. Olds, *Continued Fractions*, Random House, New York, 1963.

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