RECIPROCAL SUMS OF GENERALIZED SECOND ORDER RECURRENCE SEQUENCES

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Abstract. In this paper we extend a theorem of Hu, Sun, and Liu on reciprocal sums of second order recurrence sequences with constant coefficients to reciprocal sums of second order recurrence sequences with non-constant coefficients. Our generalization has applications to continued fractions.

1. Introduction

Let $L(E_1, E_2)$ be the set of all second order recurrence sequences $\{R_n\}_{n\in\mathbb{Z}}$ of real numbers satisfying a second order linear recurrence relation of the form

$$R_{n+1} = E_1 R_n + E_2 R_{n-1} \ (n \in \mathbb{Z}),$$

where $E_1, E_2 \in \mathbb{R}$. Let α, β be the roots of its characteristic equation

$$x^2 - E_1 x - E_2 = 0$$

with $|\alpha| < |\beta|$.

For convenience throughout, let $\mathbb{N}$ denote the set of nonnegative integers, and $\mathbb{Z}^+$ the set of positive integers.

In 2001, Hu, Sun, and Liu, [2] proved the following result.

Theorem 1.1. *Let $\{u_n\}, \{w_n\} \in L(E_1, E_2)$ with $u_0 = 0, u_1 = 1$ and let f be a function such that $f(n) \in \mathbb{Z}$ and $w_{f(n)} \neq 0$ for all $n \in \mathbb{N}$.*
1) *If $m \in \mathbb{Z}^+$, then*

$$\sum_{n=0}^{m-1} \frac{(-E_2)^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \frac{(-E_2)^{f(0)} u_{f(m)-f(0)}}{w_{f(0)} w_{f(m)}}.$$

2) *Assume that $E_1,\ E_2 \in \mathbb{R} \setminus \{0\}$ and $E_1^2 + 4E_2 \geq 0$. If $\lim_{n\to\infty} f(n) = +\infty$ and $w_1 \neq \alpha w_0$, then*

$$\sum_{n=0}^{\infty} \frac{(-E_2)^{f(n)} u_{\Delta f(n)}}{w_{f(n)} w_{f(n+1)}} = \frac{\alpha^{f(0)}}{(w_1 - \alpha w_0) w_{f(0)}},$$

where $\Delta f(n) = f(n+1) - f(n)$.

As elaborated by Hu, Sun, and Liu, interest in these identities stems from their versatile applicability in deriving a good deal of well-known identities about reciprocal sums of Fibonacci and Lucas numbers. Our first objective here is to establish an extension of the above theorem for sequences whose elements satisfy a similar second order recurrence relation but with **non-constant** coefficients. Since the set of recurrence sequences of this kind contains all sequences of numerators and denominators of convergents of simple continued fractions, our second objective is to make use of the identities so derived to deduce fruitful results

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about continued fractions. It is worth mentioning that most identities obtained here are valid over any arbitrarily base field.

Let $A := \{a_n\}_{n\in\mathbb{Z}}$ and $B := \{b_n\}_{n\in\mathbb{Z}}$ be two sequences of elements in a field and assume that $a_n \neq 0$ for all $n \in \mathbb{Z}$. Let $\mathcal{L}(A, B)$ be the set of all second order recurrence sequences $\{W_n\}_{n\in\mathbb{Z}}$ satisfying

$$W_{n+2} = b_{n+2}W_{n+1} + a_{n+2}W_n \ (n \in \mathbb{Z}). \tag{1.1}$$

The following is our main result.

Theorem 1.2. *Let $\{A_n\}, \{B_n\} \in \mathcal{L}(A, B)$, $D = A_1B_0 - A_0B_1$. For each fixed $m \in \mathbb{Z}$, define the sequence $\{C_{m,n}\}_{n\in\mathbb{Z}}$ by*

$$C_{m,0} = 0,\ C_{m,1} = 1,\quad C_{m,n} = b_{m+n}C_{m,n-1} + a_{m+n}C_{m,n-2} \ (n \in \mathbb{Z}). \tag{1.2}$$

and set

$$\alpha_m = \begin{cases} a_2a_3\cdots a_{m+1} & \text{if } m > 0,\\ 1 & \text{if } m = 0,\\ (a_1a_0a_{-1}\cdots a_{m+2})^{-1} & \text{if } m < 0.\end{cases}$$

Let f be a function from $\mathbb{N}$ to $\mathbb{Z}$. Assume that $B_{f(k)} \neq 0$ for all $k \in \mathbb{N}$.
I. If $t \in \mathbb{Z}^+$, then

$$\sum_{k=0}^{t-1} \frac{(-1)^{f(k)}\ \alpha_{f(k)}\ D\, C_{f(k),\, f(k+1)-f(k)}}{B_{f(k)}B_{f(k+1)}} = \frac{(-1)^{f(0)}\ \alpha_{f(0)}\ D\, C_{f(0),\, f(t)-f(0)}}{B_{f(0)}B_{f(t)}} = \frac{A_{f(t)}}{B_{f(t)}} - \frac{A_{f(0)}}{B_{f(0)}}.$$

II. If $\lim_{k\to\infty} f(k) = +\infty$ and $\lim_{n\to\infty} A_n/B_n = \xi$, then

$$\sum_{k=0}^{\infty} \frac{(-1)^{f(k)}\ \alpha_{f(k)}\ D\, C_{f(k),\, f(k+1)-f(k)}}{B_{f(k)}B_{f(k+1)}} = \xi - \frac{A_{f(0)}}{B_{f(0)}}.$$

Note that if we take, in Theorem 1.2,

$$a_n = E_2,\ b_n = E_1,\ B_n = w_n,\ C_{m,n} = A_n = u_n \quad (n \in \mathbb{Z}),$$

we simply recover Theorem 1.1. For another application, we recall some notation. Let $\{R_n\}_{n\in\mathbb{N}}$ be a binary recurrence defined by

$$R_{n+2} = E_1R_{n+1} + E_2R_n, \quad E_1,\ E_2 \in \mathbb{Z}\setminus\{0\}\,;\, R_0,\ R_1 \in \mathbb{Z}.$$

Suppose that $R_n \neq 0$ for all $n \in \mathbb{N}$, the characteristic polynomial $\Phi(X) := X^2 - E_1X - E_2$ is irreducible in $\mathbb{Q}[x]$ and $E_1^2 + 4E_2 > 0$. Then, for all $n \in \mathbb{N}$,

$$R_n = g_1\rho_1^n + g_2\,\rho_2^n \ \ ;\ g_1,\ g_2 \in \mathbb{Q}(\rho_1)\setminus\{0\},$$

where ρ_1 and ρ_2 are the two roots of $\Phi(X)$ so chosen that $|\rho_1| > |\rho_2|$. The following corollary gives an explicit expression of an identity in [8, Corollary 3].

Corollary 1.3. *Let $a(\neq 0)$, b, $m(\neq 0) \in \mathbb{N}$. Then*

$$\sum_{n=0}^{m-1} \frac{(-E_2)^{an}}{R_{an+b}\, R_{a(n+1)+b}} = \frac{\rho_1^{am} - \rho_2^{am}}{(\rho_1^a - \rho_2^a)\, R_b\, R_{am+b}}$$

and

$$\sum_{n=0}^{\infty} \frac{(-E_2)^{an+b}}{R_{an+b}\, R_{a(n+1)+b}} = \frac{\rho_2^b}{g_1(\rho_1^a - \rho_2^a)R_b}.$$

2. Proof of Theorem 1.2

We begin with a number of auxiliary identities involving the elements in the sequences.

Lemma 2.1. *Let $m \in \mathbb{Z}$ be fixed. For $n \in \mathbb{Z}^+$, we have*

$$C_{m-n,n} = b_{m-n+2}C_{m-n+1,n-1} + a_{m-n+3}C_{m-n+2,n-2}. \tag{2.1}$$

Proof. Taking $n = 1$ and $n = 2$, respectively, in (1.2), we get, for any $k \in \mathbb{Z}$,

$$1 = a_k C_{k-1,-1} \tag{2.2}$$

$$C_{k,2} = b_{k+2}, \tag{2.3}$$

which show that (2.1) holds for $n = 1, 2$. We proceed by induction on n. Assume (2.1) holds for n (≥ 2) and any $m \in \mathbb{Z}$. By (1.2) and the induction hypothesis, we get

$$\begin{aligned}
C_{m-n-1,n+1} &= b_m C_{m-n-1,n} + a_m C_{m-n-1,n-1} \\
&= b_m \left(b_{m-n+1}C_{m-n,n-1} + a_{m-n+2}C_{m-n+1,n-2}\right) \\
&\quad + a_m \left(b_{m-n+1}C_{m-n,n-2} + a_{m-n+2}C_{m-n+1,n-3}\right) \\
&= b_{m-n+1} \left(b_m C_{m-n,n-1} + a_m C_{m-n,n-2}\right) \\
&\quad + a_{m-n+2} \left(b_m C_{m-n+1,n-2} + a_m C_{m-n+1,n-3}\right) \\
&= b_{m-n+1}C_{m-n,n} + a_{m-n+2}C_{m-n+1,n-1}.
\end{aligned}$$

□

Lemma 2.2. *Let $m \in \mathbb{Z}$. For $n \in \mathbb{Z}^+$, we have*

$$C_{m,-n} = \frac{(-1)^{-n+1}C_{m-n,n}}{a_{m-n+2}a_{m-n+3}\cdots a_{m+1}}. \tag{2.4}$$

Proof. The assertion when $n = 1$, and any $m \in \mathbb{Z}$, corresponds exactly to (2.2). As for the case $n = 2$, putting $n = 0$ in (1.2), we get

$$C_{m,-2} = \frac{1}{a_m}\left(C_{m,0} - b_m C_{m,-1}\right) = -\frac{b_m C_{m,-1}}{a_m}.$$

Replacing b_m using (2.3) and replacing $C_{m,-1}$ using (2.2), we get

$$C_{m,-2} = -\frac{C_{m-2,2}}{a_m a_{m+1}},$$

which is the assertion when $n = 2$ and any $m \in \mathbb{Z}$. We proceed by induction on n assuming it holds for $n(\geq 2)$, and any $m \in \mathbb{Z}$. From (1.2), the induction hypothesis and Lemma 2.1, we have

$$\begin{aligned} C_{m,-n-1} &= \frac{C_{m,-n+1} - b_{m-n+1}C_{m,-n}}{a_{m-n+1}} \\ &= \frac{1}{a_{m-n+1}}\left(\frac{(-1)^{-n}C_{m-n+1,n-1}}{a_{m-n+3}\cdots a_{m+1}} - \frac{b_{m-n+1}(-1)^{-n+1}C_{m-n,n}}{a_{m-n+2}\cdots a_{m+1}}\right) \\ &= \frac{(-1)^{-n}}{a_{m-n+1}a_{m-n+2}\cdots a_{m+1}}\left(b_{m-n+1}C_{m-n,n} + a_{m-n+2}C_{m-n+1,n-1}\right) \\ &= \frac{(-1)^{-n}C_{m-n-1,n+1}}{a_{m-n+1}a_{m-n+2}\cdots a_{m+1}}. \end{aligned}$$

□

Lemma 2.3. *Let* $\{A_n\}, \{B_n\} \in \mathcal{L}(A, B)$*. Then, for* $k, l \in \mathbb{Z}$*, we have*

$$A_kB_l - A_lB_k = (-1)^l\,\alpha_l\,D\,C_{l,k-l}, \tag{2.5}$$

where D, α_l *are as defined in the statement of Theorem 1.2.*

Proof. We begin with the case $k \geq l$, i.e., $k = l + n$ for some $n \in \mathbb{N}$. Observe first that both sides of (2.5) are zero when $k = l$. There are two possibilities according to whether $l \geq 0$.
• If $l \geq 0$ is fixed, using the recurrence (1.1) we have

$$\begin{aligned} \begin{pmatrix} A_{l+1} & A_l \\ B_{l+1} & B_l \end{pmatrix} &= \begin{pmatrix} A_l & A_{l-1} \\ B_l & B_{l-1} \end{pmatrix}\begin{pmatrix} b_{l+1} & 1 \\ a_{l+1} & 0 \end{pmatrix} = \begin{pmatrix} A_{l-1} & A_{l-2} \\ B_{l-1} & B_{l-2} \end{pmatrix}\begin{pmatrix} b_l & 1 \\ a_l & 0 \end{pmatrix}\begin{pmatrix} b_{l+1} & 1 \\ a_{l+1} & 0 \end{pmatrix} \\ &= \cdots = \begin{pmatrix} A_1 & A_0 \\ B_1 & B_0 \end{pmatrix}\begin{pmatrix} b_2 & 1 \\ a_2 & 0 \end{pmatrix}\begin{pmatrix} b_3 & 1 \\ a_3 & 0 \end{pmatrix}\cdots\begin{pmatrix} b_{l+1} & 1 \\ a_{l+1} & 0 \end{pmatrix}. \end{aligned}$$

Evaluating the determinants on both sides leads to

$$A_{l+1}B_l - A_lB_{l+1} = (-1)^l\,\alpha_l\,D,$$

showing that (2.5) holds for $n = 0, 1$. Assuming the assertion up to $n - 1$, using (1.1) and the induction hypothesis, we get

$$\begin{aligned} A_{l+n}B_l - A_lB_{l+n} &= \left(b_{l+n}A_{l+n-1} + a_{l+n}A_{l+n-2}\right)B_l - A_l\left(b_{l+n}B_{l+n-1} + a_{l+n}B_{l+n-2}\right) \\ &= b_{l+n}(-1)^l\,\alpha_l\,D\,C_{l,n-1} + a_{l+n}(-1)^l\,\alpha_l\,D\,C_{l,n-2} = (-1)^l\alpha_l\,D\,C_{l,n}. \end{aligned}$$

• If $l < 0$, the assertion is proved in a similar manner noting the different shape of the α_l.

Now for the case $k < l$, we write $l = k + n$ for some $n \in \mathbb{Z}^+$. Using the result just proved above, we have

$$A_kB_l - A_lB_k = -(A_{k+n}B_k - A_kB_{k+n}) = (-1)^{k+1}\,\alpha_k\,D\,C_{k,n}. \tag{2.6}$$

To verify the assertion (2.5), it suffices to show that

$$(-1)^{k+n}\,\alpha_{k+n}\,D\,C_{k+n,-n} = (-1)^{k+1}\,\alpha_k\,D\,C_{k,n}. \tag{2.7}$$

Since from Lemma 2.2

$$C_{k+n,-n} = \frac{(-1)^{-n+1}C_{k,n}}{a_{k+2}a_{k+3}\cdots a_{k+n+1}},$$

to establish (2.7) it remains to show that

$$\alpha_k = \frac{\alpha_{k+n}}{a_{k+2}a_{k+3}\cdots a_{k+n+1}}.$$

For $k = 0$, this is trivial from its definition because

$$\alpha_0 = 1 = \frac{\alpha_n}{a_2 a_3 \cdots a_{n+1}}.$$

For $k > 0$, this is again immediate from its definition because

$$\alpha_k = a_2 a_3 \cdots a_{k+1} = \frac{\alpha_{k+n}}{a_{k+2} a_{k+3} \cdots a_{k+n+1}}.$$

For $k = -t < 0$, there are three possibilities: $-t + n > 0$, $-t + n = 0$, and $-t + n < 0$. If $-t + n > 0$, then we get from definition

$$\alpha_k = \alpha_{-t} = \frac{1}{a_1 a_0 a_{-1} \cdots a_{-t+2}} = \frac{a_2 a_3 \cdots a_{-t+n+1}}{a_{-t+2} a_{-t+3} \cdots a_{-t+n+1}} = \frac{\alpha_{k+n}}{a_{k+2} a_{k+3} \cdots a_{k+n+1}}.$$

If $-t + n = 0$, the result is clear from the definition. If $-t + n < 0$, then

$$\begin{aligned}\alpha_k = \alpha_{-t} &= \frac{1}{(a_1 a_0 a_{-1} \cdots a_{-t+n+2})(a_{-t+n+1} \cdots a_{-t+2})} \\ &= \frac{\alpha_{-t+n}}{a_{-t+n+1} a_{-t+n} \cdots a_{-t+2}} = \frac{\alpha_{k+n}}{a_{k+2} a_{k+3} \cdots a_{k+n+1}}.\end{aligned}$$

□

Now we proceed to prove Theorem 1.2. Summing a telescoping series, we get

$$\sum_{k=0}^{t-1} \left(\frac{A_{f(k+1)}}{B_{f(k+1)}} - \frac{A_{f(k)}}{B_{f(k)}} \right) = \frac{A_{f(t)}}{B_{f(t)}} - \frac{A_{f(0)}}{B_{f(0)}}.$$

Simplifying the expressions on both sides using (2.5) of Lemma 2.3, part I follows. Taking the limit as $t \to \infty$ in part I, the result of part II is immediate.

Regarding the proof of Corollary 1.3, note that both assertions follow directly from Theorem 1.2 by taking $w_n = B_n,\ f(n) = an + b$ and noting that $u_n = \frac{\rho_1^n - \rho_2^n}{\rho_1 - \rho_2}$.

3. Applications to Continued Fractions

Let $\{a_n\}_{n \in \mathbb{Z}^+}$ and $\{b_n\}_{n \in \mathbb{Z}^+}$ be two sequences of elements in a field all of whose elements are nonzero. Define the sequence $\{S_n\}_{n \in \mathbb{Z}^+}$ by

$$S_1 = \frac{a_1}{b_1}, \quad S_2 = \frac{a_1}{b_1 + \dfrac{a_2}{b_2}}$$

and generally,

$$S_n = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots + \cfrac{a_n}{b_n}}}} \quad (n \in \mathbb{Z}^+).$$

If the sequence $\{S_n\}$ converges, we write

$$\mathbf{K}(a_n/b_n) := \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \ddots}}} = \lim_{n \to \infty} S_n, \tag{3.1}$$

and call it a (non-regular) continued fraction of the element it represents. The elements a_n, b_n, are referred to as its nth **partial numerator**, respectively, nth **partial denominator**, and S_n is called the nth **approximant** (or convergent) of the continued fraction (3.1). If $a_i = 1$ and $b_i \in \mathbb{Z}^+$ $(i \geq 1)$, this is usually called a simple continued fraction, customarily denoted by $[b_1, b_2, b_3, \ldots]$. For a fixed b_0, define

$$A_{-1} = 1,\ A_0 = b_0,\ B_{-1} = 0,\ B_0 = 1, \tag{3.2}$$

and let

$$\frac{A_n}{B_n} := b_0 + S_n \quad (n \geq 1).$$

It is well-known, see [3, Chapter 2] or [6, Chapter 1], that the two sequences $\{A_n\}$ and $\{B_n\}$ satisfy the same system of second order linear recurrence relations but with different initial conditions, viz.,

$$A_n = b_n A_{n-1} + a_n A_{n-2}\ (n \geq 1), \tag{3.3}$$

$$B_n = b_n B_{n-1} + a_n B_{n-2}\ (n \geq 1). \tag{3.4}$$

The elements A_n, B_n are called its nth **numerator** and **denominator**, respectively. This recurrence is of the same form as (1.1) in Section 1. Theorem 1.2 immediately yields the following theorem.

Theorem 3.1. *Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 0}$ be two sequences in a field, all of whose elements except b_0 are nonzero. Let $\{A_n\}$, $\{B_n\}$ be two sequences satisfying* (3.2), (3.3) *and* (3.4) *and the sequence $\{C_{m,n}\}$ be as defined in* (1.2). *Let f be a function from $\mathbb{N}$ to $\mathbb{N}$. Assume that $B_{f(k)} \neq 0$ for all $k \in \mathbb{N}$.*
I. For $t \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{t-1} \frac{(-1)^{f(k)} C_{f(k),\, f(k+1)-f(k)} \prod_{i=1}^{f(k)+1} a_i}{B_{f(k)} B_{f(k+1)}} = \frac{(-1)^{f(0)} C_{f(0),\, f(t)-f(0)} \prod_{i=1}^{f(0)+1} a_i}{B_{f(0)} B_{f(t)}} = \frac{A_{f(t)}}{B_{f(t)}} - \frac{A_{f(0)}}{B_{f(0)}}.$$

II. If $\lim_{k\to\infty} f(k) = +\infty$ and if the sequence $\{A_n/B_n\}$ converges to ξ, then it is the sequence of approximants of the continued fraction representing ξ and

$$\sum_{k=0}^{\infty} \frac{(-1)^{f(k)} C_{f(k),\, f(k+1)-f(k)} \prod_{i=1}^{f(k)+1} a_i}{B_{f(k)} B_{f(k+1)}} = \xi - \frac{A_{f(0)}}{B_{f(0)}}.$$

As an application, consider the following extension of Example 4 in [6, pp. 70-71]. Define the sequences of partial denominators $\{b_n\}$, and partial numerators $\{a_n\}$, by

$$b_0 = 0,\ b_1 = 2,\ a_1 = 1,\ a_n = -\frac{n^j - (n-1)^j}{(n-1)^j - (n-2)^j},\ b_n = \frac{n^j - (n-2)^j}{(n-1)^j - (n-2)^j} \quad (n \geq 2),$$

where $j \in \mathbb{Z}^+$ is fixed. Thus, the numerators and denominators in the nth approximant of the (non-regular) continued fraction found from (3.3) and (3.4) are

$$A_{-1} = 1,\ B_{-1} = 0,\ A_0 = b_0 = 0,\ B_0 = 1,\ A_n = n^j,\ B_n = n^j + 1 \quad (n \geq 1).$$

Taking $f(n) = n$, we get $C_{f(n),f(n+1)-f(n)} = C_{n,1} = 1$ for all $n \geq 0$ and so Theorem 3.1 (I) yields for $t \geq 2$

$$\sum_{k=0}^{t-1} \frac{(k+1)^j - k^j}{(k^j+1)\left((k+1)^j+1\right)} = \frac{t^j}{t^j+1} - 0,$$

which is an obvious identity as the left-hand sum is telescoping. Both sides converge to 1, which is the value of the corresponding non-regular continued fraction.

For the case of simple continued fractions, i.e., with $a_n = 1$, Theorem 3.1 leads to the following corollary.

Corollary 3.2. *Let $\{b_n\}_{n\in\mathbb{N}}$ be a sequence in a field all of whose elements except b_0 are nonzero. Let $\{A_n\}$, $\{B_n\}$ be two sequences satisfying* (3.2), (3.3) *and* (3.4) *and the sequence $\{C_{m,n}\}$ be as defined in* (1.2). *Let f be a function from $\mathbb{N}$ to $\mathbb{N}$. Assume that $B_{f(k)} \neq 0$ for all $k \in \mathbb{N}$.*
I. For $t \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{t-1} \frac{(-1)^{f(k)}\, C_{f(k),f(k+1)-f(k)}}{B_{f(k)}B_{f(k+1)}} = \frac{(-1)^{f(0)}\, C_{f(0),f(t)-f(0)}}{B_{f(0)}B_{f(t)}} = \frac{A_{f(t)}}{B_{f(t)}} - \frac{A_{f(0)}}{B_{f(0)}}. \tag{3.5}$$

II. If $\lim_{k\to\infty} f(k) = +\infty$ and if the sequence $\{A_n/B_n\}$ converges to ξ, then it is the sequence of approximants of the continued fraction representing ξ and

$$\sum_{k=0}^{\infty} \frac{(-1)^{f(k)}\, C_{f(k),f(k+1)-f(k)}}{B_{f(k)}B_{f(k+1)}} = \xi - \frac{A_{f(0)}}{B_{f(0)}}.$$

The result of Corollary 3.2 (I) may be thought of as a generalization of equation (2.1.25) of [3, p. 26]. Both results are applicable to the case of simple continued fractions with, as well-known, the convergence being always valid. We briefly illustrate its applications now. The golden number has an appealing simple continued fraction, namely,

$$\frac{1+\sqrt{5}}{2} = [1; 1, 1, 1, \ldots].$$

Its sequences of numerators and denominators are merely two shifted Fibonacci sequences. The Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by

$$F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_{n+1} + F_n\ (n \geq 0).$$

An immediate consequence of Theorem 3.1 (II) for this particular case yields the following corollary.

Corollary 3.3. *Let g be a map from $\mathbb{Z}^+$ to $\mathbb{N}$ with $\lim_{n\to\infty} g(n) = \infty$. If $F_{g(n)+1} \neq 0$ for all $n \in \mathbb{Z}^+$, then*

$$\sum_{n=1}^{\infty} \frac{(-1)^{g(n)} F_{g(n+1)-g(n)}}{F_{g(n)+1}F_{g(n+1)+1}} = \left(\frac{1+\sqrt{5}}{2} - \frac{F_{g(1)+2}}{F_{g(1)+1}}\right).$$

As mentioned by Hu, Sun, and Liu in [2], this corollary is a host of a good deal of identities involving reciprocal sums of Fibonacci numbers. Here are some more examples.

Corollary 3.4. *Let $t \in \mathbb{Z}^+$. Then*

I. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{F_n\, F_{n+2t}} = \frac{1}{F_{2t}} \sum_{m=1}^{2t} (-1)^m \frac{F_{m+1}}{F_m},$

II. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{F_n\, F_{n+2t}} = \frac{1}{F_{2t}} \left(\sum_{m=1}^{2t} \frac{F_{m+1}}{F_m} - t(\sqrt{5}+1) \right).$

Proof. For part I, taking $m \in \{1, 2, \ldots, 2t\}$ and $g(n) = 2tn - m$ in Corollary 3.3, we get

$$\sum_{n=1}^{\infty} \frac{1}{F_{2tn-m+1} F_{2tn+2t-m+1}} = \frac{(-1)^m}{F_{2t}} \left(\frac{1+\sqrt{5}}{2} - \frac{F_{2t-m+2}}{F_{2t-m+1}} \right). \tag{3.6}$$

Summing over m from 1 to $2t$ in (3.6), we get

$$\sum_{m=1}^{2t} \sum_{n=1}^{\infty} \frac{1}{F_{2tn-m+1} F_{2tn+2t-m+1}} = \sum_{m=1}^{2t} \frac{(-1)^m}{F_{2t}} \left(\frac{1+\sqrt{5}}{2} - \frac{F_{2t-m+2}}{F_{2t-m+1}} \right),$$

and the result follows after rearranging.

For part II, multiplying by $(-1)^{m+1}$ and summing over m from 1 to $2t$ in (3.6), we get

$$\sum_{m=1}^{2t} (-1)^{m+1} \sum_{n=1}^{\infty} \frac{1}{F_{2tn-m+1} F_{2tn+2t-m+1}} = \sum_{m=1}^{2t} \frac{-1}{F_{2t}} \left(\frac{1+\sqrt{5}}{2} - \frac{F_{2t-m+2}}{F_{2t-m+1}} \right),$$

and again the result follows after rearranging. □

Remarks.

1) From the well-known Cassini formula in [4, p. 74], $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$, we deduce the following identity (26) of [7]

$$\begin{aligned} \sum_{m=1}^{2t} (-1)^m \frac{F_{m+1}}{F_m} &= \sum_{m=1}^{t} \left(\frac{F_{2m+1}}{F_{2m}} - \frac{F_{2m}}{F_{2m-1}} \right) = \sum_{m=1}^{t} \frac{F_{2m-1}F_{2m+1} - F_{2m}^2}{F_{2m}F_{2m-1}} \\ &= \sum_{m=1}^{t} \frac{1}{F_{2m-1}F_{2m}}, \end{aligned}$$

which gives another expression for the right-hand side of part I in Corollary 3.4.

2) Note that

$$\begin{aligned} \sum_{m=1}^{2t} \frac{F_{m+1}}{F_m} - t(\sqrt{5}+1) &= \sum_{m=1}^{2t} \frac{F_{m+1}}{F_m} - 2t - t(\sqrt{5}-1) \\ &= \sum_{m=1}^{2t} \left(\frac{F_{m+1}}{F_m} - 1 \right) - t(\sqrt{5}-1) = \sum_{m=1}^{2t} \frac{F_{m-1}}{F_m} - \frac{4t}{\sqrt{5}+1}, \end{aligned}$$

is the identity (3) in [1], which gives another expression for the right-hand side of part II in Corollary 3.4.

Since the function $f(n)$ is allowed to take both positive and negative integral values, the two results in Theorem 3.1, which are direct consequences of those in Theorem 1.2, remain valid if we take negative values of $f(n)$. This leads to another kind of continued fractions which has also been of much interest. We now state the result.

Theorem 3.5. *Let $A := \{a_n\}_{n\in\mathbb{Z}}$ and $B := \{b_n\}_{n\in\mathbb{Z}}$ be two sequences in a field all of whose elements are nonzero. Let $\{A_n\}, \{B_n\} \in \mathcal{L}(A, B)$ satisfy* (3.2)*, the sequence $\{C_{m,n}\}$ be as defined in* (1.2) *and let f be a map from $\mathbb{N}$ to $\mathbb{Z}$. Assume that $B_{f(k)} \neq 0$ for all $k \in \mathbb{N}$.*
I. For $t \in \mathbb{Z}^+$, we have

$$\sum_{k=0}^{t-1} \frac{(-1)^{f(k)} C_{f(k),\, f(k+1)-f(k)} \prod_{i=1}^{f(k)+1} a_i}{B_{f(k)} B_{f(k+1)}} = \frac{(-1)^{f(0)} C_{f(0),\, f(t)-f(0)} \prod_{i=1}^{f(0)+1} a_i}{B_{f(0)} B_{f(t)}} \tag{3.7}$$

$$= \frac{A_{f(t)}}{B_{f(t)}} - \frac{A_{f(0)}}{B_{f(0)}}. \tag{3.8}$$

II. If $\lim_{k\to\infty} f(k) = -\infty$ and if the sequence $\{A_{-n}/B_{-n}\}_{n\in\mathbb{Z}^+}$ converges to ψ, then

$$\sum_{k=0}^{\infty} \frac{(-1)^{f(k)} C_{f(k),\, f(k+1)-f(k)} \prod_{i=1}^{f(k)+1} a_i}{B_{f(k)} B_{f(k+1)}} = \psi - \frac{A_{f(0)}}{B_{f(0)}}.$$

As an application, consider the continued fractions whose partial quotients form an arithmetic progression. We recall the following terminology and result from [5]. Let a, b be two fixed positive integers and $\alpha = b/a$. The modified Bessel function is defined via

$$I_\nu(z) = \sum_{j=0}^{\infty} \frac{(z/2)^{\nu+2j}}{\Gamma(j+1)\Gamma(\nu+j+1)},$$

and let $U_n := I_{n+\alpha}(2/a) \quad (n \in \mathbb{Z})$. It is known that

$$U_{n-1} = (an+b)U_n + U_{n+1} \quad (n \in \mathbb{Z}).$$

Let

$$F(a,b) := b + \cfrac{1}{a+b+\cfrac{1}{2a+b+\cfrac{1}{3a+b+\ddots}}} = [b,\ a+b,\ 2a+b,\ 3a+b, \cdots].$$

Lehmer, [5], showed that

$$F(a,b) = \frac{U_{-1}}{U_0} = \frac{I_{\alpha-1}(2/a)}{I_\alpha(2/a)}.$$

We now give another explicit expression for this result. In Theorem 3.5, taking

$$a_n = 1,\ b_{n+1} = an+b,\ \alpha = b/a,\ A_n = U_n,\ B_n = U_{n-1},\ f(n) = -n \quad (n \in \mathbb{Z}),$$

we see that $C_{f(n), f(n+1)-f(n)} = C_{-n,-1} = 1$ and

$$\frac{U_{-m}}{U_{-m-1}} \sim \frac{1}{a(-m+\alpha+1)} \to 0 \qquad (m \to \infty).$$

Hence, assuming that $U_n \neq 0$ for all $n < 0$, we have

$$\sum_{j=0}^{m-1} \frac{(-1)^j}{U_{-j-1}U_{-j-2}} = \frac{U_{-m}}{U_{-m-1}} - \frac{U_0}{U_{-1}}$$

and, provided $I_{\alpha-n} \neq 0 \quad (n \geq 1)$, we discover the identity

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{I_{-j-1+\alpha}(2/a) I_{-j-2+\alpha}(2/a)} = -\frac{1}{F(a,b)}.$$

As our final remark, let us mention that through the use of Auric's Theorem, Theorem 10 on page 207 and also Corollary 11 on page 209 of [6], another kind of result can be obtained, namely, let $\{X_n\}_{n \geq -1}$ satisfy the recurrence relation

$$X_n = b_n X_{n-1} + a_n X_{n-2} \quad (n \geq 1),$$

with $a_n \neq 0$ and $X_n \neq 0$ for all n. If

$$\sum_{n=0}^{\infty} \frac{\prod_{m=1}^{\infty}(-a_m)}{X_{n-1} X_n} = \infty,$$

then the continued fraction $\mathbf{K}(a_n/b_n)$ converges to the finite value $-X_0/X_{-1}$, and

$$\frac{A_k}{B_k} + \frac{X_0}{X_{-1}} \sim C \left(\sum_{n=0}^{k} \frac{\prod_{m=1}^{n}(-a_m)}{X_{n-1} X_n} \right)^{-1} \qquad (n \to \infty)$$

for some constant $C \neq 0$.

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