ON THE PERIODICITY OF CERTAIN RECURSIVE SEQUENCES

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ABSTRACT. In 2000, Viswanath showed that random Fibonacci sequences grow exponentially and calculated the rate at which they grow assuming the coin flipped was fair. In this paper, we explore the Fibonacci sequences generated by finite, repeating sequences of pluses and minuses. The main results of this paper will be to show the necessary conditions for a sequence to be periodic, as well as to show all the possible periods of the sequences. It will be clear that the set of periodic random Fibonacci sequences is a subset of measure 0 of random Fibonacci sequences.

In his 2000 paper, Viswanath [4] examined the growth properties of random Fibonacci sequences generated with a fair coin flip; his paper resulted in a new mathematical constant, 1.13198824.... His result was a specific, computed case of an earlier result due to Furstenberg and Kesten [2]. Here we will examine the properties of the subset of random sequences that are periodic. We will show that these sequences can only occur when the plus minus sequence generated by the coin flips, which corresponds to a Bernoulli sequence, is itself periodic. This, of course, happens with probability 0, but we will study the properties of these finite sequences of pluses and minuses acting on the integers.

Suppose we have a recursion, R_1 defined as

$$R_1 : X_{n+1} = X_{n-1} + X_n.$$

We can think of X_{n+1} as a vector,

$$\overrightarrow{X}_{n+1} = \begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} = \begin{pmatrix} X_{n-1} + X_n \\ X_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix}.$$

Let

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

then

$$\overrightarrow{X}_{n+1} = M_1 \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix}.$$

Similarly, if we define a second recursion, R_2 as

$$R_2 : X_{n+1} = X_{n-1} - X_n,$$

we have

$$\overrightarrow{X}_{n+1} = \begin{pmatrix} X_{n+1} \\ X_n \end{pmatrix} = \begin{pmatrix} X_{n-1} - X_n \\ X_n \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix}.$$

Letting

$$M_2 = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix},$$

Research conducted under the guidance of Professor Pat McDonald, while at New College of Florida.

we have

$$\overrightarrow{X}_{n+1} = M_2 \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix}.$$

Finally, let $\Omega = \{\omega \mid \omega = \omega_1\omega_2 \cdots \omega_n \cdots \ \omega_i = 1 \text{ or } \omega_i = 2\}$, then we have that $\Omega \simeq \{f \mid f : \mathbb{N} \to \{1,2\}\}$.

Definition 1. *We say $\{X_i\}_{i=1}^{\infty}$ is a random Fibonacci sequence if*

(1) $f \in \Omega$
(2) $\overrightarrow{X}_n = M_{f(n)}\overrightarrow{X}_{n-1}$

i.e.

$$\overrightarrow{X}_n = \left(\prod_{i=n}^{1} M_{f(i)}\right)\overrightarrow{X}_1.$$

Note that because of the way this is defined here we needed to have i decreasing. In general, this won't be the case.

Definition 2. *With M_1, M_2 as before, and $f \in \Omega$, we say σ is a motif if $\sigma = \prod_{i=1}^{n} M_{f(i)}$. We will use $|\sigma| = n$ to denote the length of the motif, which is the number of pluses and minuses, or equivalently, the number of matrices multiplied together.*

Definition 3. *Suppose σ is a motif, then σ is periodic if there exists a minimal $k \in \mathbb{N}$ such that $\sigma^k = I$, the identity matrix. We will use $p(\sigma) = k$ as notation. Let $\mathcal{M}$ be the collection of all periodic motifs.*

Example 1. *Consider the motif*

$$\sigma = M_2M_1 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

which is one plus, followed by one minus. We see that $(M_1M_2)^6 = I$, and since 6 is the smallest number to do this, $p(\sigma) = 6$.

It will prove useful later to have another version of this example. We can look at the seed vector to this sequence, $\binom{x}{y}$ and explicitly list the terms in the sequence until we see the periodicity. The result is

$$\underbrace{x, y}, x+y, -x, y, -x-y, -x, -y, -x+-y, x, -y, x+y, \underbrace{x, y} \ldots$$

and we see that σ is periodic with period 6 because we have the seeds repeated at the end of σ after 6 repetitions.

Proposition 1. *Suppose σ is a motif that gives a bounded sequence, then the sequence is periodic.*

Proof. Since the sequence of integers is bounded, we can plot the sequence on a finite integer lattice. The sequence has two seeds, corresponding to a point in the lattice, (x_0, y_0). As the sequence progresses, we move around the integer lattice, arriving at (x_n, y_n) on the nth step. Let $|\sigma| = m$. By the pigeon hole principle, there exists a k such that $(x_k, y_k) = (x_0, y_0)$ where $k \equiv 0 \pmod{m}$, as required. We note further that $p(\sigma)|k$. $\square$

Proposition 2. *Suppose $\sigma \in \mathcal{M}, f \in \Omega$, and $\sigma = \prod_{i=1}^{n} M_{f(i)}$, then if*

$$\tau = \left(\prod_{i=l}^{n} M_{f(i)}\right)\left(\prod_{j=1}^{l-1} M_{f(j)}\right),$$

we have $\tau \in \mathcal{M}$. This simply means that if σ is periodic, then the periodicity is independent of the starting point within the motif.

Proof. For simplicity of notation, we will let $p(\sigma) = 1$; it will be clear that the proof holds for $p(\sigma) > 1$. Let $\prod_{i=l}^{n} M_{f(i)} = A$ and $\prod_{j=1}^{l-1} M_{f(j)} = B$, then we see that $\sigma = BA$ and $\tau = AB$. Since $p(\sigma) = 1, BA = I$, and we need to show $\tau = AB = I$. Since $\det(M_{f(k)}) = \pm 1$ for all k, we see that $\det(B) = \pm 1$, therefore, B^{-1} exists. Thus, we have $A = B^{-1}I = IB^{-1}$, thus, $AB = I$ as required. If $p(\sigma) = k > 1$, then we have $\sigma^k = \underbrace{AB \ldots AB}_{ktimes} = I$ and must simply repeat the argument to obtain the result. $\square$

Theorem 1. *Suppose $\sigma \in M$, $p(\sigma) = k$ and $|\sigma| = m$ then $mk \equiv 0 \pmod 3$.*

Proof. We revisit Example 1 for a hint on how to prove this. In the example, we let $\sigma = \{+, -\}$ and let it act on x, y. The result was

$$x, y, x+y, -x, y, -x-y, -x, -y, -x+-y, x, -y, x+y, x, y \ldots$$

which showed us periodicity with period 6. If we look at this same example, but don't combine the terms after each step, we see the first few terms are

$$x, y, x+y, y-x-y, x+y+y-x-y, y-x-y+x-y-y+x+y \ldots$$

What this shows us is that at the nth step, we have $\mathrm{F}(n-2)$ x terms and $\mathrm{F}(n-1)$ y terms, where $\mathrm{F}(n)$ is the standard Fibonacci sequence. This pattern clearly holds for general σ.

Using this, we see that in order for the sequence to be periodic, we need x, y to repeat at step $n, n+1$, thus, we need two consecutive odd Fibonacci numbers so all but one x and all but one y can cancel each other out. Furthermore, we require that when the number of x's is odd, the number of y's be even, and vice versa. That is, if $mk - 1 = n$, where $|\sigma| = m$ and $p(\sigma) = k$, we need

$$F(mk) = F(n-1) \equiv 1 \pmod 2 \text{ and } F(mk) = F(n-2) \equiv 0 \pmod 2$$

$$F(mk-1) = F(n-2) \equiv 1 \pmod 2 \text{ and } F(mk-1) = F(n-1) \equiv 0 \pmod 2$$

By the odd, odd, even structure of the Fibonacci sequence, if $F(n-1) \equiv 0 \pmod 2$, $F(n-2) \equiv 1 \pmod 2$, then $n \equiv 1 \pmod 3$. Thus, we need $mk \equiv 0 \pmod 3$. Furthermore, at step $mk+1$, we need $F(n-1) \equiv 1 \pmod 2$ and $F(n-2) \equiv 0 \pmod 2$. This gives $n \equiv 2 \pmod 3$. Thus, since $mk - 1 = n \equiv 2 \pmod 3$, $mk \equiv 0 \pmod 3$ as required. $\square$

Lemma 1. *Suppose $\sigma \in \mathcal{M}$, $p(\sigma) = k$, then there exists $\tau \in \mathcal{M}$ such that $p(\tau) = d(k)$, where $d(k)$ is a divisor of k, for all $d(k)$.*

Proof. By the hypotheses, σ is periodic, with period k. Suppose $|\sigma| = m$ and $d|k$. We can see that if we take the motif σ^k, it is periodic of period 1. Suppose further we take the motif σ^d. We can see that if $dl = k$, then $(\sigma^d)^l = \sigma^k$. Thus, σ^d is periodic of period l. $\square$

In the coming 2 lemmas, we will need the following identity. Suppose A is a 2×2 matrix with positive integer entries, and $A^4 = I$. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then standard matrix multiplication tells us that

$$A^4_{2,2} = a^2bc + bcd^2 + 2abcd + b^2c^2 + 2bc^3 + c^4 = 1. \tag{1}$$

Lemma 2. *Suppose $\sigma \in \mathcal{M}$, and $\sigma^4 = I$. If the eigenvalues of σ are ± 1, then $p(\sigma) = 2$.*

Proof. Let

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the characteristic equation, $\lambda^2 - 1 = 0$ causes $\det(\sigma) = ad - bc = -1$ and $a = -d$. Using these identities, and matrix multiplication, we have

(1) $\sigma^2_{1,1} = a^2 + bc = a^2 + 1 - a^2 = 1$
(2) $\sigma^2_{1,2} = ab + bd = b(a + d) = b(0) = 0$
(3) $\sigma^2_{2,1} = ac + cd = c(a + d) = c(0) = 0$
(4) $\sigma^2_{2,2} = 1 - a^2 + c^2$.

It remains to show that the fourth item is equal to 1 by showing that $a^2 = c^2$. We do this by using Equation (1) along with our derived identities.

$$\sigma^4_{2,2} = a^2bc + bcd^2 + 2abcd + b^2c^2 + 2bc^3 + c^4 = 1$$
$$\sigma^4_{2,2} = (1 - a^2) + 2(1 - a^2)c^2 + c^4 = 1.$$

The only solutions to this are $a = \pm c$ and $a = \sqrt{c^2 - 2}$. Since $\sqrt{c^2 - 2} \notin \mathbb{Z}$, we need only concern ourselves with $a = \pm c$, which gives us $a^2 = c^2$. Therefore, $\sigma_{2,2} = 1$, completing the proof that $p(\sigma) = 2$. □

Lemma 3. *Suppose $\sigma \in \mathcal{M}$, then $p(\sigma) \neq 4$.*

Proof. By Lemma 2, we know that if the eigenvalues of σ are $\pm$ 1, then the period is 2, so we consider the only other case here. Assume for a contradiction that $p(\sigma) = 4$. Suppose the eigenvalues of σ are $\pm i$. Thus, the characteristic equation is $\lambda^2 + 1 = 0$, and consequently, we have $a = -d$ and $\det(\sigma) = ad - bc = 1$. This gives us the useful identity $bc = -(a^2 + 1)$. Exploiting (1) again, we reduce as we did before.

$$\sigma^4_{2,2} = a^2bc + bcd^2 + 2abcd + b^2c^2 + 2bc^3 + c^4 = 1$$
$$\sigma^4_{2,2} = -a^2(a^2 + 1) - a^2(a^2 + 1) + 2a^2(a^2 + 1) + (a^2 + 1)^2 - 2c^2(a^2 + 1) + c^4 = 1$$
$$\sigma^4_{2,2} = (a^2 + 1)^2 - 2c^2(a^2 + 1) + c^4 = 1.$$

Again we have $a = \pm c$ with the $a = \sqrt{c^2 - 2}$ thrown out. Consequently, if $p(\sigma) = 4$, and the eigenvalues are $\pm i$, we have 2 cases:

(1) $\sigma = \begin{pmatrix} a & b \\ a & -a \end{pmatrix}$
(2) $\sigma = \begin{pmatrix} a & b \\ -a & -a \end{pmatrix}$

To show neither of these cases are valid, consider the parity of our matrices M_1 and M_2. Then we have

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that the parity of M_1^3 is the identity matrix, which is also clear for M_2. Thus, we have only 3 options for the parity of any matrix representation of a motif, namely

$$M_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_1^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{ and } M_1^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We see now that using the parity argument, if $p(\sigma) = 4$, then $\sigma_{1,1}^4 = \sigma_{2,1}^4 = \sigma_{2,2}^4$, but that doesn't occur in any of our three possibilities. This contradicts the assumption that $p(\sigma) = 4$, thus completing the proof. □

Theorem 2. *If $\sigma \in \mathcal{M}, p(\sigma) = k$, then k is a divisor of 6.*

Proof. To prove this, we will examine the eigenvalues of σ. Suppose $p(\sigma) = k$, then $\sigma^k = I$, therefore, if λ_1, λ_2 are the eigenvalues of σ, then $\lambda_1^k = \lambda_2^k = 1$, since the eigenvalues of I are both 1. This tells us that the eigenvalues of σ must be two kth roots of unity. Furthermore, we see that if $\lambda_1 = a+bi$, then $\lambda_2 = a-bi$ since $\mathrm{tr}(\sigma) \in \mathbb{Z}$. Furthermore, we have $\mathrm{tr}(\sigma) = 2a$. But we see $|a| \leq 1$, therefore, $|a| = 1$, or $|a| = \frac{m}{2}$ where $m \in \mathbb{Z}$. If $\lambda_1 = \lambda_2 = \pm 1$, we have that $k = 2$. We examine further the case where $|a| = \frac{m}{2}$.

With polar coordinates, we have $\lambda_1 = e^{i\theta}$ and we know $\cos(\theta) = a = \frac{m}{2}$, but $m \in \mathbb{Z}$, thus the only valid solution for (m, θ) is $(-1, \frac{2\pi}{3})$ and $(1, \frac{\pi}{3})$. This gives us $k = 3$ and $k = 6$, respectively. Notice that if k is a prime greater than 3, or if $k = 9$, we do not have the proper conditions for the eigenvalues. Namely, the lack of the $\theta = \frac{2\pi}{3}$ ray in the geometric interpretation of the roots of unity. Also, by Lemmas 2 and 3, we have no motifs of order 4. However, if we multiply 3 by powers of 2, we see that we maintain the required rays, thus we have the possibility of $k = 3(2^n)$ for some n. However, this leads to a contradiction by Lemmas 1 and 3. Namely, for $n \geq 2$, 4 is a divisor by Lemma 1, but this isn't possible by Lemma 3.

We have exemplified periods of 1, 2, 3, and 6, and it is clear that for all $k > 6$, one of the following is true:

(1) k is prime
(2) k has a prime divisor greater than 3
(3) $4|k$
(4) $9|k$

We have seen that in any of these cases, a period of length k is impossible, thus completing the proof. □

The following is an alternate, shorter proof to this theorem.

Proof. It is well-known that in the group $\mathrm{GL}(2, \mathbb{Z})$, any element of finite order has order 1, 2, 3, 4, or 6 as seen in Lemma 2.11 of [3]. Let $\mathfrak{M} =< M_1, M_2 >$ be the group of motifs under standard matrix multiplication. It is clear that $\mathfrak{M} \leq \mathrm{GL}(2, \mathbb{Z})$. Thus, all elements of finite order in $\mathfrak{M}$ have order 1, 2, 3, 4, or 6. By Lemma 3, no motif of order 4 exists. Computations show the existence of motifs of order 1, 2, 3, and 6. □

Within the first proof of this theorem is most of the proof concerning the elements of finite order of $\mathrm{GL}(2, \mathbb{Z})$, so the alternate proof is just a slimmed down version for those already familiar with the fact. In conclusion, we see that if a motif gives rise to a periodic sequence, then the period of that sequence must be a divisor of 6, and 3 divides the product of the length of the motif and the period. An interesting corollary to these results is the generation of the infinite group $\mathfrak{M}$ of matrices generated by M_1 and M_2. In $\mathfrak{M}$, if an element has finite order, that order is a divisor of 6, otherwise, the element has infinite order. It can be seen that $\mathfrak{M} \leq \mathrm{GL}(2, \mathbb{Z})$, and further research may reveal a stronger connection, or new facts altogether.

References

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