ON THE CONSTRUCTION OF A FAMILY OF ALMOST POWER FREE SEQUENCES

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Abstract. We introduce the concept of an integer sequence to be almost power free and show that the primorial plus one sequence $2 \cdot 3 \cdot 5 \cdots p_n + 1$ and its generalizations are almost power free. In addition a stronger result is also proven, namely that the primorial plus one sequence is free from perfect powers.

1. Introduction

A frequently occurring question in the theory of integer sequences is the existence or non-existence of perfect powers, natural numbers of the form m^n where $m, n \in \mathbb{N}\backslash\{1\}$, among the terms of a given integer sequence. Such questions have proved difficult to resolve, in connection with a number of well-known integer sequences, such as the Fibonacci and Lucas sequences. Indeed, using a novel approach that combines the theory of logarithmic forms with the modular method, M. Mignotte et al [1] has recently shown that the only Fibonacci numbers which are perfect powers are $F_0 = 0$, $F_1 = F_2 = 1$, $F_6 = 8$ and $F_{12} = 144$. The Fibonacci sequence is a particular example of a broader class of sequences we shall define here as being "almost power free", in that for each integer $s \in \mathbb{N}\backslash\{1\}$, there can only be at most finitely many perfect powers in the sequence having exponent s. With this definition in mind, it is natural to question whether there are other familiar, but non-trivial examples of sequences, such as the Fibonacci sequence, which satisfy the "almost power free" condition. In this paper, we shall construct such a family of sequences using a generalization of the primorial plus one sequence that is, $2 \cdot 3 \cdot 5 \cdots p + 1$, found in Euclid's proof for the infinitude of primes. In addition, we shall also demonstrate that the sequence $2 \cdot 3 \cdot 5 \cdots p + 1$, is in point of fact free from all perfect powers.

2. Main Results

Before establishing the main result, let us first make precise the idea of a sequence being almost power free with the following definition.

Definition 2.1. *A sequence of positive integers $\{a_n\}$ is said to be* **almost power free***, if for each integer $s \in \mathbb{N}\backslash\{1\}$ there exists an $m_s \in \mathbb{N}$ such that for all $n \geq m_s$, there does not exist an $N \in \mathbb{N}$ such that $a_n = N^s$.*

In what follows the set of prime numbers is denoted by P.

Theorem 2.2. *Suppose $\{a_n\}$ is a sequence of positive integers defined in the following manner. Partition the set $P\backslash\{2,5\} = \bigcup_{i=1}^{\infty} A_i$, where each set A_i is finite with $A_i \cap A_j = \emptyset$, for $i \neq j$, and let $a_n = \prod_{p \in A_n} p$. Then the associated sequence $\{b_n\}$, defined by $b_n = 2 \cdot 5 \cdot a_1 a_2 \cdots a_n + 1$, is an almost power free sequence.*

Proof. We argue via proof by contradiction. Assume for any exponent $s \in \mathbb{N}\backslash\{1\}$ there exists two infinite subsequences $\{n_k\}$, $\{N_k\}$ of positive integers greater than unity, such that $b_{n_k} = N_k^s$. Furthermore, we may assume without loss of generality that s is prime, since if $s = rp$, where $r, p \in \mathbb{N}$ and p is prime, then $N_k^s = (N_k^r)^p$. Clearly as b_{n_k} is odd so must N_k, hence $N_k \equiv \pm 1, \pm 3$ or 5 (mod 10). First note that $N_k \not\equiv 5 \pmod{10}$, since the contrary would imply that $5|b_{n_k}$, which is impossible. Now as $b_{n_k} \equiv 1 \pmod{10}$, if $N_k \equiv \pm 3 \pmod{10}$ then the only positive integer powers of N_k congruent to $1 \pmod{10}$ are N_k^{4n}, for $n = 1, 2, \ldots$, thus as p is prime the equality $b_{n_k} = N_k^p$ is impossible and so $N \not\equiv \pm 3 \pmod{10}$. Similarly, if $N \equiv -1 \pmod{10}$, then as the only positive integer powers of N_k congruent to $1 \pmod{10}$ are N_k^{2n}, for $n = 1, 2, \ldots$, we need only examine the equality $b_{n_k} = (10m - 1)^2$, where $m \in \mathbb{N}$. Upon expanding and rearranging terms, one finds that

$$a_1 a_2 \cdots a_{n_k} = 10m^2 - 2m,$$

which is impossible as the right-hand side is even, while the left-hand side is odd, thus $N_k \not\equiv -1 \pmod{10}$. Alternatively, if $N_k \equiv 1 \pmod{10}$ then, a similar argument establishes the impossibility of the equality $b_{n_k} = (10m + 1)^2$. Hence, as all other positive integer powers of N_k are congruent to $1 \pmod{10}$, we are left to consider the remaining equality $b_{n_k} = (10m+1)^p$, where p is an odd prime. Upon expanding and rearranging terms one finds that

$$a_1 a_2 \cdots a_{n_k} = 10^{p-1}m^p + \binom{p}{1}10^{p-2}m^{p-1} + \binom{p}{2}10^{p-3}m^{p-2} + \cdots + \binom{p}{p-1}m. \tag{2.1}$$

As every prime $p|\binom{p}{i}$, for $i = 1, 2, \ldots, p-1$, we deduce from (2.1) that $p \neq 5$ since otherwise 5 would divide the right-hand side but not the left-hand side of (2.1). Thus assume p is an odd prime other than 5. Now by construction the product $a_1 a_2 \cdots a_{n_k}$ is square free and for k sufficiently large $p|a_1 a_2 \cdots a_{n_k}$. Consequently, $p \nmid m$ since otherwise p^2 would divide the left-hand side of (2.1), thus $p \nmid 10^{p-1}m^p$ and so cannot divide the right-hand side of (2.1), thus producing the final contradiction and so $N_k \not\equiv 1 (\mathrm{mod}\, 10)$. Hence, the original assumption is false and so for n sufficiently large there cannot exist, for each fixed exponent $p \in \mathbb{N}\backslash\{1\}$, an $N \in \mathbb{N}$ such that $b_n = N^p$. Thus the sequence $\{b_n\}$ must be almost power free. □

If p_n denotes the n-th prime, then a simple inductive argument reveals that the set partition $P\backslash\{2, 5\} = \bigcup_{i=1}^{\infty} A_i$ given by $A_1 = \{3\}$ and $A_i = \{p_{i+2}\}$, for $i > 1$, gives rise to the sequence $b_n = 2 \cdot 3 \cdot 5 \cdots p_{n+2} + 1$. Thus from Theorem 2.2 we conclude that the primorial plus one sequence $2 \cdot 3 \cdot 5 \cdots p_n + 1$ must be almost power free. To conclude we prove using a modification of the proof of Theorem 2.2 the following stronger result.

Theorem 2.3. *The primorial plus one sequence given by* $b_n = 2 \cdot 3 \cdot 5 \cdots p_n + 1$ *is power free.*

Proof. As $b_1 = 3$ and $b_2 = 7$ are not perfect powers, we consider the sequence b_n where $n > 2$. Fix n and again assume without loss of generality that for a prime exponent p, there exists an $N \in \mathbb{N}\backslash\{1\}$, such that $b_n = N^p$. Clearly as b_n is odd so must N, hence $N \equiv \pm 1, \pm 3$ or 5 (mod 10). First note that $N \not\equiv 5 \pmod{10}$, since the contrary would imply that $5|b_n$, which is impossible. Now as $b_n \equiv 1 \pmod{10}$, if $N \equiv \pm 3 \pmod{10}$ then the only positive integer powers of N congruent to $1 \pmod{10}$ are N^{4s}, for $s = 1, 2, \ldots$. Thus, as p is prime the equality $b_n = N^p$ is impossible and so $N \not\equiv \pm 3 \pmod{10}$. Similarly, if $N \equiv -1 \pmod{10}$, then as the only positive integer powers of N congruent to $1 \pmod{10}$ are N^{2s}, for $s = 1, 2, \ldots$, we need only examine the equality $b_n = (10m-1)^2$, where $m \in \mathbb{N}$. Upon expanding and rearranging terms, one finds that

$$3 \cdot 7 \cdot 11 \cdots p_n = 10m^2 - 2m,$$

which is impossible as the right-hand side is even, while the left-hand side is odd, thus $N \not\equiv -1 (\text{mod}\, 10)$. Alternatively, if $N \equiv 1 \pmod{10}$ then a similar argument establishes the impossibility of the equality $b_n = (10m+1)^2$. Hence, as all other positive integer powers of N are congruent to $1 \pmod{10}$, we are left to consider the remaining equality $b_n = (10m+1)^p$, where p is an odd prime. Upon expanding and rearranging terms one finds that

$$3 \cdot 7 \cdot 11 \cdots p_n = 10^{p-1}m^p + \binom{p}{1}10^{p-2}m^{p-1} + \binom{p}{2}10^{p-3}m^{p-2} + \cdots + \binom{p}{p-1}m. \tag{2.2}$$

As every prime $p|\binom{p}{i}$, for $i = 1, 2, \ldots, p-1$, we deduce from (2.2) that $p \neq 5$ since otherwise 5 would divide the right-hand side but not the left-hand side of (2.2). Thus assume p is an odd prime other than 5. We now show that p divides the left-hand side but not the right-hand side of (2.2). First note from the equality $b_n = (10m+1)^p = N^p$ that $(N, p_i) = 1$, for all $i = 1, 2, \ldots, n$, and so $N > p_n$, for if $N \leq p_n$ then at least one of the p_i must divide N. Consequently $b_n = p_1p_2 \cdots p_n + 1 = N^p > p_n^p$ and so $p_1p_2 \cdots p_n \geq p_n^p$, but this can only be true if $n > p$. However, as $p_n > n$ we deduce that $p_n > p$ and as $p \neq 2, 5$ one must have $p|3 \cdot 7 \cdot 11 \cdots p_n$. Furthermore, as the left-hand side of (2.2) is square free, we note $p \nmid m$ since otherwise p^2 would divide the right-hand side of (2.2). Thus, $p \nmid 10^{p-1}m^p$ but $p|\binom{p}{i}$, for $i = 1, 2, \ldots, p-1$, and so p cannot divide the right-hand side of (2.2), a clear and final contradiction and so $N \not\equiv 1 \pmod{10}$. Hence the original assumption is false and thus the sequence $b_n = 2 \cdot 3 \cdot 5 \cdots p_n + 1$ must be power free. □

References

[1] Y. Bugeaud, M. Mignotte and S. Siksek, *Classical and Modular Approaches to Exponential Diophantine Equations. I. Fibonacci and Lucas Perfect Powers*, Ann. of Math, **163** (2006), 969–1018.

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