SOME PERIODICITIES IN THE CONTINUED FRACTION EXPANSIONS OF FIBONACCI AND LUCAS DIRICHLET SERIES

CHRIS K. CALDWELL AND TAKAO KOMATSU

Abstract. In this paper we consider the Fibonacci Zeta functions $\zeta_F(s)=\sum_{n=1}^{\infty}F_n^{-s}$ and the Lucas Zeta functions $\zeta_L(s)=\sum_{n=0}^{\infty}L_n^{-s}$. The sequences $\{A_\nu\}_{\nu\ge 0}$ and $\{B_\nu\}_{\nu\ge 0}$, which are derived from $\sum_{\nu=1}^{n}F_\nu^{-s}=A_n/B_n$, satisfy certain recurrence formulas. We examine some properties of the periodicities of A_n and B_n. For example, let m and k be positive integers. If $n\ge mk$, then $B_n\equiv 0 \pmod{F_k^m}$ (with a similar result holding for A_n). The power of 2 which divides B_n is $\lfloor \frac{n}{6}\rfloor+\sum_{i=0}^{\infty}\lfloor\frac{n}{3\cdot 2^i}\rfloor$.

1. Introduction

Consider the so-called *Fibonacci and Lucas Zeta functions*:

$$\zeta_F(s)=\sum_{n=1}^{\infty}\frac{1}{F_n^s}, \qquad \zeta_L(s)=\sum_{n=0}^{\infty}\frac{1}{L_n^s}.$$

In [13] the analytic continuation of these series is discussed. In [4] it is shown that the numbers

$$\zeta_F(2),\zeta_F(4),\zeta_F(6) \quad (\text{respectively}, \zeta_L(2),\zeta_L(4),\zeta_L(6))$$

are algebraically independent, and that each of

$$\zeta_F(2s) \quad (\text{respectively}, \zeta_L(2s)) \qquad (s=4,5,6,\dots)$$

may be written as a rational (respectively, algebraic) function of these three numbers over $\mathbb{Q}$, e.g.

$$\zeta_F(8)-\frac{15}{14}\zeta_F(4)=\frac{1}{378(4u+5)^2}\Big(256u^6-3456u^5+2880u^4+1792u^3v$$
$$-11100u^3+20160u^2v-10125u^2+7560uv+3136v^2-1050v\Big),$$

where $u=\zeta_F(2)$ and $v=\zeta_F(6)$. Similar results are obtained in [4] for the alternating sums

$$\zeta_F^*(2s):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{F_n^{2s}} \quad \left(\text{respectively}, \zeta_L^*(2s):=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{L_n^{2s}}\right) \quad (s=1,2,3,\dots).$$

From the main theorem in [5] it follows that for any positive distinct integers s_1,s_2,s_3 the numbers $\zeta_F(2s_1)$, $\zeta_F(2s_2)$, and $\zeta_F(2s_3)$ are algebraically independent if and only if at least one of s_1,s_2,s_3 is even. Other types of algebraic independence, including the functions

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$$\sum_{n=1}^{\infty} \frac{1}{F_{2n-1}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{F_{2n}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n-1}^s}, \quad \sum_{n=1}^{\infty} \frac{1}{L_{2n}^s}$$

are discussed in [9].

On the other hand, in [8] Fibonacci zeta functions and Lucas zeta functions including

$$\zeta_F(1), \zeta_F(2), \zeta_F(3), \zeta_F^*(1), \zeta_L(1), \zeta_L(2), \zeta_L^*(1)$$

are expanded as non-regular continued fractions whose components are Fibonacci or Lucas numbers. For example, [8, Theorem 1] says

Lemma 1.1. *We have*

$$\zeta_F(1) = \cfrac{1}{F_2 - \cfrac{F_1^2}{F_3 - \cfrac{F_2^2}{F_4 - \cfrac{F_3^2}{\ddots - \cfrac{F_{n-1}^2}{F_{n+1} - \dots}}}}}$$

and

$$\sum_{\nu=1}^{n} \frac{1}{F_\nu} = \frac{A_n}{B_n},$$

where $\{A_\nu\}_{\nu\geq 0}$ *and* $\{B_\nu\}_{\nu\geq 0}$ *are determined by the recurrence formulas:*

$$\begin{array}{llll} A_\nu = F_{\nu+1}A_{\nu-1} - F_{\nu-1}^2 A_{\nu-2} & (\nu \geq 2), & A_0 = 0, & A_1 = 1; \\ B_\nu = F_{\nu+1}B_{\nu-1} - F_{\nu-1}^2 B_{\nu-2} & (\nu \geq 2), & B_0 = 1, & B_1 = 1. \end{array}$$

Similar continued fraction expansions with corresponding recurrence relations hold for $\zeta_F(2)$, $\zeta_F(3)$, $\zeta_F^*(1)$, $\zeta_L(1)$, $\zeta_L(2)$, $\zeta_L^*(1)$ and related Fibonacci and Lucas Dirichlet series [8, Table 1]. In [8, Theorem 5.1] the periodicity of the sequences $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ modulo t for any integer $t \geq 2$ is considered using a result recently obtained in [7].

Lemma 1.2. *Let* $t \geq 2$ *be any integer, and let* $\{Y_n\}_{n\geq 0}$ *be a sequence of integers satisfying the recurrence relation*

$$Y_\nu = T(\nu)Y_{\nu-1} + U(\nu)Y_{\nu-2} \qquad (\nu \geq 2)$$

with sequences $\{T(\nu)\}_{\nu\geq 2}$ *and* $\{U(\nu)\}_{\nu\geq 2}$ *of integers, which are periodic modulo* t*. Then the sequence* $\{Y_n\}_{n\geq 0}$ *is ultimately periodic modulo* t*. If* $U(\nu) = 1$ *for all* $\nu \geq 2$*, then the sequence* $\{Y_n\}_{n\geq 0}$ *is periodic modulo* t*.*

By applying this lemma to the recurrence formulas for A_n and B_n in [8, Table 1], the following result is obtained in [8, Theorem 5.2].

Lemma 1.3. *For any integer* $t \geq 2$*, the sequences* $(A_n)_{n\geq 0}$ *and* $(B_n)_{n\geq 0}$ *are ultimately periodic modulo* t*.*

However, the exact period has not been known. In this paper we discuss the details about periodicity. For example, let m and k be positive integers. If $n \geq mk$, then $B_n \equiv 0 \pmod{F_k^m}$ (with a similar result holding for A_n). The power of 2 which divides B_n is $\left\lfloor \frac{n}{6} \right\rfloor + \sum_{i=0}^{\infty} \left\lfloor \frac{n}{3\cdot 2^i} \right\rfloor$.

2. Fibonacci-type Zeta Functions

Consider Fibonacci-type numbers $\{G_n\}_{n\ge 1}$ defined by

$$G_n = G_{n-1} + G_{n-2} \quad (n \ge 2)$$

with positive integral initial values G_1 and G_2. Let

$$\zeta_G(s) = \sum_{n=1}^{\infty} \frac{1}{G_n^s}\,, \qquad \zeta_G^*(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{G_n^s}.$$

Continued fraction expansions of $\zeta_G(s)$ and $\zeta_G^*(s)$ are obtained in [8, Lemma 2, Lemma 3]. Namely,

$$\zeta_G(s) = \cfrac{1}{G_1^s - \cfrac{G_1^{2s}}{G_1^s + G_2^s - \cfrac{G_2^{2s}}{G_2^s + G_3^s - \cfrac{G_3^{2s}}{G_3^s + G_4^s - \ddots - \cfrac{G_{n-1}^{2s}}{G_{n-1}^s + G_n^s - \cdots}}}}}$$

and

$$\zeta_G^*(s) = \cfrac{1}{G_1^s + \cfrac{G_1^{2s}}{-G_1^s + G_2^s + \cfrac{G_2^{2s}}{-G_2^s + G_3^s + \cfrac{G_3^{2s}}{-G_3^s + G_4^s + \ddots + \cfrac{G_{n-1}^{2s}}{-G_{n-1}^s + G_n^s + \cdots}}}}}.$$

Now A_n (respectively B_n) are defined as the numerator (respectively denominator) convergent of the continued fraction expansion given for $\zeta_G(s)$:

$$\frac{A_n}{B_n} = \cfrac{1}{G_1^s - \cfrac{G_1^{2s}}{G_1^s + G_2^s - \cfrac{G_2^{2s}}{G_2^s + G_3^s - \cfrac{G_3^{2s}}{G_3^s + G_4^s - \ddots - \cfrac{G_{n-1}^{2s}}{G_{n-1}^s + G_n^s}}}}}.$$

Hence, $\{A_\nu\}_{\nu\ge 0}$ and $\{B_\nu\}_{\nu\ge 0}$ satisfy the following recurrence formulas.

$$A_\nu = (G_{\nu-1}^s + G_\nu^s)A_{\nu-1} - G_{\nu-1}^{2s}A_{\nu-2} \qquad (\nu \ge 2), \qquad A_0 = 0, \qquad A_1 = 1;$$
$$B_\nu = (G_{\nu-1}^s + G_\nu^s)B_{\nu-1} - G_{\nu-1}^{2s}B_{\nu-2} \qquad (\nu \ge 2), \qquad B_0 = 1, \qquad B_1 = G_1^s.$$

In fact, A_ν and B_ν can be expressed explicitly as follows.

Lemma 2.1. *For $n = 1, 2, \ldots$*

$$A_n = (G_1 G_2 \ldots G_n)^s \sum_{\nu=1}^{n} \frac{1}{G_\nu^s}, \qquad B_n = (G_1 G_2 \ldots G_n)^s.$$

Proof. By induction we have $B_n = (G_1G_2 \dots G_n)^s$. Thus,

$$A_n = B_n \sum_{\nu=1}^{n} \frac{1}{G_\nu^s} = (G_1G_2 \dots G_n)^s \sum_{\nu=1}^{n} \frac{1}{G_\nu^s}.$$

□

Similarly, if A_n^* (respectively B_n^*) are defined as the numerator (respectively denominator) convergent of the continued fraction expansion given for $\zeta_G^*(s)$, then $\{A_\nu^*\}_{\nu\ge 0}$ and $\{B_\nu^*\}_{\nu\ge 0}$ satisfy the following recurrence formulas.

$$\begin{aligned} A_\nu^* &= (-G_{\nu-1}^s + G_\nu^s)A_{\nu-1}^* + G_{\nu-1}^{2s}A_{\nu-2}^* \qquad (\nu \ge 2), \qquad A_0^* = 0, \qquad A_1^* = 1;\\ B_\nu^* &= (-G_{\nu-1}^s + G_\nu^s)B_{\nu-1}^* + G_{\nu-1}^{2s}B_{\nu-2}^* \qquad (\nu \ge 2), \qquad B_0^* = 1, \qquad B_1^* = G_1^s. \end{aligned}$$

Similar to Lemma 2.1, we have the following.

Lemma 2.2. *For* $n = 1, 2, \dots$

$$A_n^* = (G_1G_2 \dots G_n)^s \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{G_\nu^s}, \qquad B_n^* = (G_1G_2 \dots G_n)^s.$$

Some reciprocal sums of consecutive Fibonacci or Lucas numbers have been studied (e.g. [1, 14]). For example, the reciprocal sum of $G_n^sG_{n+1}^s$ has the following continued fraction expansion.

Corollary 2.3.

$$\sum_{n=1}^{\infty} \frac{1}{G_n^sG_{n+1}^s} = \cfrac{1}{G_1^sG_2^s - \cfrac{G_1^{2s}G_2^s}{G_1^s + G_3^s - \cfrac{G_2^sG_3^s}{G_2^s + G_4^s - \cfrac{G_3^sG_4^s}{G_3^s + G_5^s - \ddots - \cfrac{G_{n-1}^sG_n^s}{G_{n-1}^s + G_{n+1}^s - \dots}}}}},$$

Proof. By [8, Lemma 2.1], we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{G_n^s G_{n+1}^s}\\
&=\cfrac{G_1^{-s}G_2^{-s}}{1-\cfrac{G_2^{-s}G_3^{-s}}{G_1^{-s}G_2^{-s}+G_2^{-s}G_3^{-s}-\cfrac{G_1^{-s}G_2^{-s}G_3^{-s}G_4^{-s}}{G_2^{-s}G_3^{-s}+G_3^{-s}G_4^{-s}-\cfrac{G_2^{-s}G_3^{-s}G_4^{-s}G_5^{-s}}{G_3^{-s}G_4^{-s}+G_4^{-s}G_5^{-s}-\dots}}}}\\
&=\cfrac{1}{G_1^sG_2^s-\cfrac{G_1^sG_3^{-s}}{G_1^{-s}G_2^{-s}+G_2^{-s}G_3^{-s}-\cfrac{G_1^{-s}G_2^{-s}G_3^{-s}G_4^{-s}}{G_2^{-s}G_3^{-s}+G_3^{-s}G_4^{-s}-\cfrac{G_2^{-s}G_3^{-s}G_4^{-s}G_5^{-s}}{G_3^{-s}G_4^{-s}+G_4^{-s}G_5^{-s}-\dots}}}}\\
&=\cfrac{1}{G_1^sG_2^s-\cfrac{G_1^{2s}G_2^s}{G_3^s+G_1^s-\cfrac{G_4^{-s}}{G_2^{-s}G_3^{-s}+G_3^{-s}G_4^{-s}-\cfrac{G_2^{-s}G_3^{-s}G_4^{-s}G_5^{-s}}{G_3^{-s}G_4^{-s}+G_4^{-s}G_5^{-s}-\dots}}}}\\
&=\cfrac{1}{G_1^sG_2^s-\cfrac{G_1^{2s}G_2^s}{G_1^s+G_3^s-\cfrac{G_2^sG_3^s}{G_4^s+G_2^s-\cfrac{G_5^{-s}}{G_3^{-s}G_4^{-s}+G_4^{-s}G_5^{-s}-\dots}}}}.
\end{aligned}
$$

□

3. Fibonacci Zeta Functions

Let $G_1 = G_2 = 1$. Then $G_n = F_n$ are reduced to Fibonacci numbers. Consider the continued fraction expansion

$$
\zeta_F(s)=\cfrac{1}{F_1^s-\cfrac{F_1^{2s}}{F_1^s+F_2^s-\cfrac{F_2^{2s}}{F_2^s+F_3^s-\cfrac{F_3^{2s}}{\ddots-\cfrac{F_{n-1}^{2s}}{F_{n-1}^s+F_n^s-\dots}}}}}
$$

and

$$
\sum_{\nu=1}^{n}\frac{1}{F_\nu^s}=\frac{A_n}{B_n},
$$

where $\{A_\nu\}_{\nu\ge 0}$ and $\{B_\nu\}_{\nu\ge 0}$ are determined by the recurrence formulas:

$$A_\nu = (F_{\nu-1}^s + F_\nu^s)A_{\nu-1} - F_{\nu-1}^{2s}A_{\nu-2} \qquad (\nu \ge 2), \qquad A_0 = 0, \qquad A_1 = 1;$$
$$B_\nu = (F_{\nu-1}^s + F_\nu^s)B_{\nu-1} - F_{\nu-1}^{2s}B_{\nu-2} \qquad (\nu \ge 2), \qquad B_0 = 1, \qquad B_1 = F_1^s.$$

Theorem 3.1. *Let m be a positive integer.*

(1) *For all $n \ge (\lceil m/s \rceil + 1)k$, $A_n \equiv 0 \pmod{F_k^m}$.*
(2) *For all $n \ge \lceil m/s \rceil k$, $B_n \equiv 0 \pmod{F_k^m}$.*

Remark. *If $s = 1$ and $m = 1$, then this matches Theorem 5.3 in* [8].

Proof. The second assertion follows from the fact $F_k|F_{ik}$ $(i = 1, 2, \ldots)$ (see e.g. [10, Theorem 16.1]) and by Lemma 2.1 $B_n = F_1^sF_2^s\ldots F_n^s$. On the other hand,

$$F_k^m \left| \left(\frac{F_1F_2\cdots F_n}{F_\nu}\right)^s\right.$$

holds for $n \ge \lceil m/s \rceil k$ if $k \nmid \nu$, and for $n \ge (\lceil m/s \rceil + 1)k$ if $k|\nu$. $\square$

Next, we shall consider the periodicity modulo the power of a prime p. We may assume $s = 1$ because the general case is obtained by multiplying by s. Hence, $B_n = F_1F_2\cdots F_n$ and $A_n = (F_1F_2\cdots F_n)\sum_{\nu=1}^n F_\nu^{-1}$.

Proposition 3.2. *The power of 2 which divides B_n is $\left\lfloor \frac{n}{6}\right\rfloor + \sum_{i=0}^\infty \left\lfloor \frac{n}{3\cdot 2^i}\right\rfloor$.*

Then, we have the following.

Theorem 3.3. *Let m be a positive integer, and choose $\sigma = \sigma(n)$ to be the least positive integer for which $m \le \left\lfloor \frac{\sigma}{6}\right\rfloor + \sum_{i=0}^\infty \left\lfloor \frac{\sigma}{3\cdot 2^i}\right\rfloor$. Then $B_{\sigma-1} \not\equiv 0 \pmod{2^m}$ and $B_k \equiv 0 \pmod{2^m}$ for all $k \ge \sigma$.*

The proof of Proposition 3.2 depends upon the following facts.

Lemma 3.4.

$$2\|F_{6m-3} \qquad (m = 1, 2, \ldots).$$

For $d = 3, 4, \ldots$

$$2^d\|F_{3\cdot 2^{d-2}(2m-1)} \qquad (m = 1, 2, \ldots).$$

We need some more lemmas in order to prove Lemma 3.4. The proof without these lemmas can be achieved by applying [6, Corollary 1].

Lemma 3.5. ([11], [10, Theorem 35.5])
The period of the generalized Fibonacci sequence modulo 2^n is $3 \cdot 2^{n-1}$.

Lemma 3.6. ([2], [10, Ex.35.42, Ex.35.43])
For $n \ge 1$

$$F_{3\cdot 2^n} \equiv 2^{n+2} \pmod{2^{n+3}},$$
$$L_{3\cdot 2^n} \equiv 2 + 2^{n+2} \pmod{2^{n+4}}.$$

Lemma 3.7. *For* $n = 1, 2, \ldots$

$$2 \nmid L_{3n\pm1}, \quad 2\|L_{6n}, \quad 2^2\|L_{6n-3}.$$

Proof. Note that

$$\{L_n\}_{n\ge1} = 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, \ldots.$$

The first assertion is clear because L_{3n-2} and L_{3n-1} are odd and L_{3n} is even. Since

$$\{L_n \pmod{2)^3}\}_{n\ge1} = \underbrace{1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, 2}_{12}, \underbrace{1, 3, 4, 7, 3, 2, 5, 7, 4, 3, 7, 2}_{12}, \ldots$$

and by Lemma 3.5 the period of the Lucas sequence modulo 2^3 is 12, we have the second and the third assertions. □

Proof of Lemma 3.4. By Lemma 3.5 the period of the Fibonacci sequence modulo 2^2 is 6. Since

$$\{F_n \pmod{2)^2}\}_{n\ge1} = \underbrace{1, 1, 2, 3, 1, 0}_{6}, \underbrace{1, 1, 2, 3, 1, 0}_{6}, \ldots,$$

we have $2\|F_{6m-3}$ $(m = 1, 2, \ldots)$.

Next, we shall show that $F_{9\cdot2^{d-2}} \equiv 2^d \pmod{2^{d+1}}$ $(d = 3, 4, \ldots)$. For $d = 3$, $F_{18} = 2584 \equiv 8 \pmod{16}$. By Lemma 3.7 we have $L_{9\cdot2^{d-2}} \equiv 2 \pmod 4$ or $L_{9\cdot2^{d-2}} = 2+4k'$ for some integer k'. Assume that for some integer $l(\ge 3)$

$$F_{9\cdot2^{l-2}} \equiv 2^l \pmod{2^{l+1}}$$

or there exists an integer k such that $F_{9\cdot2^{l-2}} = 2^l + k \cdot 2^{l+1}$. By $F_{2n} = F_nL_n$ (see e.g. [10, Corollary 5.5 (5.13)]),

$$\begin{aligned} F_{9\cdot2^{l-1}} &= F_{9\cdot2^{l-2}}L_{9\cdot2^{l-2}} = (2^l + k\cdot2^{l+1})(2+4k') \\ &= 2^{l+1} + (k + k' + 2kk')\cdot 2^{l+2}. \end{aligned}$$

Hence, $F_{9\cdot2^{l-1}} \equiv 2^{l+1} \pmod{2^{l+2}}$. By induction, we have $F_{9\cdot2^{d-2}} \equiv 2^d \pmod{2^{d+1}}$ $(d = 3, 4, \ldots)$.

Since by Lemma 3.5 the period of the Fibonacci sequence modulo 2^{d+1} is $3\cdot2^d$, we also have

$$2^d\|F_{9\cdot2^{d-2}+(m-1)\cdot3\cdot2^d} = F_{3\cdot2^{d-2}(4m-1)} \quad (m = 1, 2, \ldots).$$

Similarly, by $2^d\|F_{3\cdot2^{d-2}}$, we have

$$2^d\|F_{3\cdot2^{d-2}+(m-1)\cdot3\cdot2^d} = F_{3\cdot2^{d-2}(4m-3)} \quad (m = 1, 2, \ldots).$$

Therefore, we have the desired results. □

Proof of Proposition 3.2. Similarly to Lucas numbers in Lemma 3.7, $2 \nmid F_n$ if and only if $3 \nmid n$. By Lemma 3.4 the power of 2 which divides F_n is 1 if $3|k$ and $6 \nmid k$, and is 2 more than the power of 2 dividing n if $6|n$. □

We need additional notation in order to address the cases where p is an odd prime. Let $\nu_p(r)$ denote the exponent of the highest power of a prime p which divides r. Namely, $p^{\nu_p(r)}\|r$, or $p^{\nu_p(r)}|r$ and $p^{\nu_p(r)+1} \nmid r$. $\nu_p(F_n)$ and $\nu_p(L_n)$ are characterized in [12]. Let f_p denote the first positive index for which $p|F_{f_p}$. This index is called the *rank of apparition* or *Fibonacci entry-point* of p. For example, $f_2 = 3$, $f_3 = 4$, $f_5 = 5$, $f_7 = 8$, $f_{11} = 10$, $f_{13} = 7$ and $f_{17} = 9$.

The *order* of p in F_{f_p} is denoted by $e_p = \nu_p(F_{f_p})$. Namely, $p^{e_p}|F_{f_p}$ and $p^{e_p+1} \nmid F_{f_p}$. Note that usually $e_p = 1$ and no exceptional prime has been found for $p < 10^{14}$ [3].

Lemma 3.8. [12] *Let p be an odd prime with $p \neq 5$. If $f_p|n$, then $\nu_p(F_n) = \nu_p(n) + e_p$. If $f_p \nmid n$, then $\nu_p(F_n) = 0$.*

Corollary 3.9. *The power of an odd prime p which divides $B_n = F_1F_2 \cdots F_n$ is*

$$(e_p - 1)\left\lfloor \frac{n}{f_p} \right\rfloor + \sum_{i=0}^{\infty} \left\lfloor \frac{n}{f_p p^i} \right\rfloor .$$

Theorem 3.10. *Let m be a positive integer, and choose $\sigma = \sigma(n)$ to be the least positive integer for which $m \leq (e_p - 1)\left\lfloor \frac{\sigma}{f_p} \right\rfloor + \sum_{i=0}^{\infty} \left\lfloor \frac{\sigma}{f_p p^i} \right\rfloor$. Then $B_{\sigma-1} \not\equiv 0 \pmod{p^m}$ and $B_n \equiv 0 \pmod{p^m}$ for all $n \geq \sigma$.*

Example. *For all $n \geq 5m$, $B_n \equiv 0 \pmod{5^{\sum_{i=0}^{\infty}\lfloor m/5^i \rfloor}}$. It is easy to see this. Since $f_5 = 5$, $5^{e_5}|F_5 = 5$, so $e_5 = 1$. Setting $\sigma = 5m$ yields the result.*

Proposition 3.11. *The power of a prime p which divides $A_{f_p p^m}$ is*

$$e_p(p^m - 1) + \frac{p^m - 1}{p - 1} - m.$$

Proof. When $n = f_p p^m$ in Corollary 3.9, we see that the power of p which divides $B_{f_p p^m}$ is

$$(e_p - 1)p^m + (p^m + \cdots + p + 1) = e_p p^m + \frac{p^m - 1}{p - 1}.$$

(Note that $(e_p - 1)p^m$ is usually 0.) By Lemma 3.8 we know $p^{m+e_p} \| F_{f_p p^m}$, and $p^{m+e_p} \nmid F_i$ $(1 \leq i < f_p p^m)$. Thus, the power of p which divides the denominator of $\sum_{i=1}^{f_p p^m} F_i^{-1}$, when written as a single reduced fraction, is p^{m+e_p}. Since $A_{f_p p^m} = B_{f_p p^m} \sum_{i=1}^{f_p p^m} F_i^{-1}$, we have the desired result. ☐

4. Lucas Zeta Functions

Consider the continued fraction expansion

$$\zeta_L(s) = \cfrac{1}{L_1^s - \cfrac{L_1^{2s}}{L_1^s + L_2^s - \cfrac{L_2^{2s}}{L_2^s + L_3^s - \cfrac{L_3^{2s}}{\ddots - \cfrac{L_{n-1}^{2s}}{L_{n-1}^s + L_n^s - \dots}}}}}$$

and

$$\sum_{\nu=1}^{n} \frac{1}{L_\nu^s} = \frac{A_n}{B_n},$$

where $\{A_\nu\}_{\nu \geq 0}$ and $\{B_\nu\}_{\nu \geq 0}$ are determined by the recurrence formulas:

$$A_\nu = (L_{\nu-1}^s + L_\nu^s)A_{\nu-1} - L_{\nu-1}^{2s}A_{\nu-2} \quad (\nu \geq 2), \quad A_0 = 0, \quad A_1 = 1;$$

$$B_\nu = (L_{\nu-1}^s + L_\nu^s)B_{\nu-1} - L_{\nu-1}^{2s}B_{\nu-2} \quad (\nu \geq 2), \quad B_0 = 1, \quad B_1 = L_1^s.$$

Theorem 4.1. *Let m be a positive integer.*

(1) *For all $n \geq (2\lceil m/s \rceil + 1)k$, $A_n \equiv 0 \pmod{L_k^m}$.*
(2) *For all $n \geq (2\lceil m/s \rceil - 1)k$, $B_n \equiv 0 \pmod{L_k^m}$.*

Proof. The second statement in Theorem 4.1 is based upon the fact $L_k | L_{(2i-1)k}$ $(i = 1, 2, \dots)$(see e.g. [10, Theorem 16.6]) and $B_n = L_1^s L_2^s \dots L_n^s$. □

Lemma 4.2. [12] *Let p be an odd prime with $p \neq 5$. If the period modulo p of the Fibonacci sequences is not equal to $4f_p$ and $n \equiv f_p/2 \pmod{f_p}$, then $\nu_p(L_n) = \nu_p(n) + e(p)$. Otherwise, $\nu_p(L_n) = 0$.*

Next, we shall consider the periodicity modulo the power of a prime p. We may assume $s = 1$ without loss of generality. Hence, $B_n = L_1 L_2 \cdots L_n$ and $A_n = (L_1 L_2 \cdots L_n) \sum_{\nu=1} L_\nu^{-1}$.

Proposition 4.3. *The power of 2 which divides B_n is $2\left\lfloor \frac{n}{3} \right\rfloor - \left\lfloor \frac{n}{6} \right\rfloor$.*

Proof. It is clear from Lemma 3.7. □

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Department of Mathematics and Statistics, University of Tennessee at Martin, Martin, TN 38238-0002
E-mail address: caldwell@utm.edu

Graduate School of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan
E-mail address: komatsu@cc.hirosaki-u.ac.jp