GENERALIZATIONS OF VOSMANSKY'S IDENTITY

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Abstract. We first use the Fundamental Theorem of Algebra to give an almost immediate proof of the identity

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n}\binom{x+2n-k}{2n} = (-1)^n\binom{x}{n}\binom{x+n}{n} \tag{0.1}$$

valid for all complex values of x and all non-negative integers n. The identity was found by J. Vosmansky when x is a non-negative integer, and proved, in this case, by L. Carlitz. We then generalize, and prove, that for any integer r, and any complex x.

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n+r}\binom{x+2n-k}{2n+r} = (-1)^n\binom{2n}{n}\binom{x+n}{2n+r}\frac{\binom{x+n}{n+r}}{\binom{x+n}{n}}. \tag{0.2}$$

In fact we prove more generally that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{n+r}\binom{y+n-k}{n+r} \tag{0.3}$$

$$= (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k+r}\binom{y}{n-k+r} \tag{0.4}$$

$$= (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k+r}\binom{y+n-k}{n-k+r} \tag{0.5}$$

valid for all complex x and y and any integer r and for any integer $n \geq 0$.

1. Introduction: Vosmansky's Identity for Non-negative Integers

Vosmansky [6] discovered the identity

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{r+k}{2n}\binom{r+2n-k}{2n} = (-1)^n\binom{r}{n}\binom{r+n}{n} \tag{1.1}$$

where r and n are n-negative integers. A proof using generating functions with two variables was given by Carlitz [2], who considered the summation

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{r+k}{2n}\binom{s+2n-k}{2n} \tag{1.2}$$

where r and s are integers. He obtained the generating function

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}R(n,k,r)x^n y^k = (x-y)^r(1-x)^{-r-1}(1-y)^{-r-1} \tag{1.3}$$

where he defined

$$R(n,k,r)=\sum_{j=0}^{r}(-1)^j\binom{r}{k}\binom{n+j}{r}\binom{k+r-j}{r} \tag{1.4}$$

from which he was able to establish (1.1). Carlitz's derivation takes up two pages. In Part 3 of [2] Kaucky showed that (1.1) implies Dixon's remarkable identity

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3=(-1)^n\binom{2n}{n}\binom{3n}{n}. \tag{1.5}$$

Remark 1.1. *While we are mentioning Dixon's Identity, we would like to remind the reader of two similar identities, both of which are found in [3], see Equations (3.81) and (6.6), respectively.*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}=(1-1)^n=0 \tag{1.6}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2=(-1)^n\binom{2n}{n}. \tag{1.7}$$

The reader should notice that the power of the binomial coefficients in the sums of Equations (1.5) through (1.7) does not exceed three. This is because N. G. de Brujin [1] used asymptotic methods to show that <u>*no*</u> *closed form exists for* $\sum_{k=0}^{n}(-1)^k\binom{n}{k}^a$*, whenever* $a\geq 4$*. De Brujin's observation explains why all the identities in this paper involve summands that have at most a product of three binomial coefficients.*

In the present paper, we offer two generalizations of Identity (1.1). The first generalization, Theorem 2.1, extends the Identity (1.1) to all complex x. The second generalization, Theorem 2.2, extends the results of Theorem 2.1. Theorem 2.2 introduces an additional parameter in the convolution pair. Both of these generalizations are natural extensions of Identity (1.1) and provide insight into the generating function proof provided in [2].

2. Two Generalizations of Identity (1.1)

Our first generalization extends Identity (1.1) to all complex x. This extension relies on the observation that $\binom{x}{n}$ is a polynomial in x of degree n. Once we interpret the binomial coefficients as polynomials, we are able to apply the Fundamental Theorem of Algebra to Identity (1.1). In particular, we have the following theorem.

Theorem 2.1. *For all complex* x *and integers* $n\geq 0$

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n}\binom{x+2n-k}{2n}=(-1)^n\binom{x}{n}\binom{x+n}{n}. \tag{2.1}$$

Proof of Theorem 2.1. We recall that $\binom{x}{n}$ is a polynomial of degree n in x. Then it is clear that the right side of (2.1) is a polynomial of degree $2n$ in x, whereas at first glance the left side appears to be of degree $4n$ in x. However, the left side is also the $2n$th difference of a polynomial of degree $4n$, and the sum is actually a polynomial of degree $2n$. A corollary to the Fundamental Theorem of Algebra states that two polynomials of the same degree are identically equal if they are equal for more distinct values than their degree. Hence, in order to establish (2.1), it will be sufficient to exhibit $2n+1$ distinct values of x for which the identity (2.1) is true.

The relation is trivially true for $n = 0$. Let $n \geq 1$. It is clear that both sides of (2.1) are equal to 0 for the 2n distinct values $x = 0, 1, 2, ...n-1$ and $x = -1, -2, ... -n$. Also, when $x = n$ the relation

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{n+k}{2n}\binom{3n-k}{2n} = (-1)^n\binom{2n}{n}$$

is trivially true since every term in the series is zero except when $k = n$. Therefore we have equality in (2.1) for $2n+1$ distinct values of x and so the relation is true for all complex x. □

A natural generalization of Identity (2.1) occurs by replacing $\binom{x+k}{2n}\binom{x+2n-k}{2n}$ with $\binom{x+k}{2n+1}\binom{x+2n-k}{2n+1}$. Gould analyzed this generalization as far back as 1953 [4]. In particular, he used a Fundamental Theorem of Algebra type proof, similar to that of Theorem 2.1, to show

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n+1}\binom{x+2n-k}{2n+1} = (-1)^n\frac{x}{n+1}\binom{2n}{n}\binom{x+n}{2n+1}. \tag{2.2}$$

Remark 2.1. *Identity (2.2) is equivalent to Formula (6.38) of [3], where, for $m \geq 1$, Formula (6.38) is best written as*

$$\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{x+k}{2m}\frac{k}{x+k}\binom{x+2m-k}{2m}\frac{2m-k}{x+2m-k}$$

$$= \frac{(-1)^m}{2}\binom{x}{m}\binom{x+m}{m}\frac{x}{x+m} = \frac{(-1)^m}{2}\binom{2m}{m}\binom{x+m}{2m}\frac{x}{x+m} = \frac{(-1)^m}{2m!^2}\prod_{j=0}^{m-1}(x^2-j^2).$$

With the advent of computer algebra, we are able to analyze the more general situation $\binom{x+k}{2n+r}\binom{x+2n-k}{2n+r}$, where r is integral. Note that when $r = 0$, we obtain Identity (2.1). In particular, we show the following theorem.

Theorem 2.2. *For any integral r and any complex x,*

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n+r}\binom{x+2n-k}{2n+r} = (-1)^n\binom{2n}{n}\binom{x+n}{2n+r}\frac{\binom{x+n}{n+r}}{\binom{x+n}{n}}. \tag{2.3}$$

Proof of Theorem 2.2. The proof of Theorem 2.2 utilizes the Creative Telescoping Algorithm of Zeilberger [5]. Assume r is a nonzero integer. The case of $r = 0$ is Theorem 2.1. Let

$$f(n) = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n+r}\binom{x+2n-k}{2n+r}. \tag{2.4}$$

Apply the ct package [5] on *Maple* to show that

$$\frac{-2(2n+1)(x+n+1)(x-n-r)}{(r+2n+1)(r+n+1)(2n+r+2)}f(n) = f(n+1). \tag{2.5}$$

If we let

$$g(n) = (-1)^n\binom{2n}{n}\binom{x+n}{2n+r}\frac{\binom{x+n}{n+r}}{\binom{x+n}{n}}, \tag{2.6}$$

which is the right side of (2.3), we can show via elementary factorial manipulations that $g(n)$ satisfies

$$\frac{-2(2n+1)(x+n+1)(x-n-r)}{(r+2n+1)(r+n+1)(2n+r+2)}g(n) = g(n+1). \tag{2.7}$$

Hence, both $g(n)$ and $f(n)$ obey the same recurrence relation. Moreover, since $f(0) = g(0)$, we conclude that $f(n) = g(n)$. $\square$

Remark 2.2. *Recall that the Gamma Function allows us to define the notion of* $r!$*, where* r *is a complex number, except a negative integer. Then Equations (2.5) and (2.7), and hence Theorem 2.2, are valid for any complex number* r*.*

3. Two Variable Generalizations of Vosmanski's Sum

Encouraged by Theorem 2.2, we made more experiments with *Maple*, and are able to offer a two variable generalization of Vosmansky's sum.

Theorem 3.1. *For all complex* x *and* y *and* r *and for any integer* $n \geq 0$*,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{n+r}\binom{y+n-k}{n+r} \tag{3.1}$$

$$= (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k+r}\binom{y}{n-k+r} \tag{3.2}$$

$$= (-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k+r}\binom{y+n-k}{n-k+r}. \tag{3.3}$$

When n *is replaced by* $2n$*, and* $x = y$*, then all three sums are evaluated in closed form by formula (2.3) in Theorem (2.2).*

Proof of Theorem 3.1. In a manner similar to Theorem 2.2, the proof relies on the ct package of *Maple* [5]. In particular, Zeilberger's algorithm shows that

$$(n+r+2)(n+r+1)(n+2r+2)X(n+2) + (2n+2r+3)(n+r+1)(y-x)X(n+1)$$
$$- (n+y++x+2)(n+1)(n-y-x+2r)X(n) = 0,$$

where $X(n)$ is any one of the three sums in Equations (3.1) through (3.3). Then, by checking any two initial conditions, say $X(0)$ and $X(1)$, we prove the theorem. $\square$

In Equations (3.1) and (3.2), we let $r = 0$ and, $n \to 2n$, to obtain an identity that involves a convolution on both the upper and lower indices.

Corollary 3.1.

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n}\binom{y+2n-k}{2n} = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x}{k}\binom{y}{2n-k} \tag{3.4}$$

valid for all complex values of x *and* y *and all non-negative integers* n*.*

Not only this, but we have a second result that involves the convolution in both upper and lower summation indices, which we record as Corollary 3.2. We obtain Corollary 3.2 by letting $r = 0$ and $n \to 2n$ in Equations (3.1) and (3.3).

Corollary 3.2.

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{k}\binom{y+2n-k}{2n-k} = \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x}{k}\binom{y}{2n-k} \tag{3.5}$$

valid for all complex values of x and y and all non-negative integers n.

Alternative Proof of Corollary 3.2. The proof of Corollary 3.2 is done easily without use of the computer by using Corollary 3.1 and the well-known relation

$$\binom{-x}{k} = (-1)^k\binom{x+k-1}{k}. \tag{3.6}$$

We have, using (3.4),

$$\begin{aligned}
&\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{k}\binom{y+2n-k}{2n-k}\\
&= \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{-x-1}{k}\binom{-y-1}{2n-k}\\
&= \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{-x-1+k}{2n}\binom{-y-1+2n-k}{2n}, \quad \textit{by } (3.5)\\
&= \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+2n-k}{2n}\binom{y+k}{2n}, \quad \textit{by } (3.4)\\
&= \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x+k}{2n}\binom{y+2n-k}{2n}\\
&= \sum_{k=0}^{2n}(-1)^k\binom{2n}{k}\binom{x}{k}\binom{y}{2n-k}, \quad \textit{by } (3.5).
\end{aligned}$$

□

Remark 3.1. *When $x = y = 2n$, all three convolution sums reduce to Dixon's alternating sum of binomial cubes*

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^3 = (-1)^n\binom{2n}{n}\binom{3n}{n}. \tag{3.7}$$

4. Future Research Projects

We conclude this paper by briefly mentioning three research problems related to the theorems in Sections 2 and 3.

Problem 1: Is it possible to provide combinatorial proofs of Theorems 2.2 and 3.1?

Problem 2: What are the $q-$analogs to Theorems 2.2 and 3.1? Do these $q-$analogs have combinatorial meaning?

Problem 3: Are there other two or three variable generalizations of Identity 1.1?

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