A PROOF OF A CONJECTURE OF MELHAM

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ABSTRACT. In this paper, we consider Melham's conjecture involving Fibonacci and Lucas numbers. After rewriting it in terms of Fibonomial coefficients, we give a solution of the conjecture by evaluating a certain q-sum using contour integration.

1. INTRODUCTION

The Fibonacci numbers are defined for $n > 0$ by

$$F_{n+1} = F_n + F_{n-1}$$

where $F_0 = 0$ and $F_1 = 1$.

The Lucas numbers are defined for $n > 0$ by

$$L_{n+1} = L_n + L_{n-1}$$

where $L_0 = 2$ and $L_1 = 1$.

The Binet forms of the Fibonacci and Lucas sequences are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad \text{and} \qquad L_n = \alpha^n + \beta^n$$

where α, β are $(1 \pm \sqrt{5})/2$.

Recently as an interesting generalization of the binomial coefficients, the Fibonomial coefficients have taken the interest of several authors. For their properties, we refer to [1, 6, 8, 9, 11, 14, 15, 17, 19, 21, 22]. The Fibonomial coefficient is defined, for $n \geq m \geq 1$, by

$$\begin{Bmatrix} n \\ m \end{Bmatrix} = \frac{F_1 F_2 \cdots F_n}{(F_1 F_2 \cdots F_{n-m})(F_1 F_2 \cdots F_m)} = \frac{F_n!}{F_{n-m}! F_m!}$$

with $\begin{Bmatrix} n \\ n \end{Bmatrix} = \begin{Bmatrix} n \\ 0 \end{Bmatrix} = 1$ where F_n is the nth Fibonacci number and $F_n! = F_1 F_2 \cdots F_n$ is nth Fibonacci factorial.

These coefficients satisfy the relation:

$$\begin{Bmatrix} n \\ m \end{Bmatrix} = F_{m+1} \begin{Bmatrix} n-1 \\ m \end{Bmatrix} + F_{n-m-1} \begin{Bmatrix} n-1 \\ m-1 \end{Bmatrix}. \tag{1.1}$$

In [8], Hoggatt considered Fibonomial coefficients with indices in arithmetic progressions. For example, he defined the following generalization by taking F_{kn} instead of F_n for a fixed positive integer k:

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_k = \frac{F_k F_{2k} \cdots F_{kn}}{(F_k F_{2k} \cdots F_{k(n-m)})(F_k F_{2k} \cdots F_{km})}.$$

The Fibonomials appear in several places in the literature; we give two examples: the $n \times n$ right-adjusted Pascal matrix P_n whose (i, j) entry is given by

$$(P_n)_{ij} = \binom{j-1}{j+i-n-1}.$$

Carlitz [2] was the first to link the Fibonomial coefficients with the auxiliary polynomial of the right-adjusted Pascal matrix P_n via its coefficients. Since then, some relationships between the generalized Pascal matrix and the Fibonomial coefficients have been constructed, and the Fibonomial coefficients have been studied by some authors [2, 3, 12, 13, 20].

Secondly, in [5, 7, 8, 10], one can find that the nth powers of Fibonacci numbers satisfy the following auxiliary polynomial

$$C_n(x) = \sum_{i=0}^{n} (-1)^{i(i+1)/2} \left\{ \begin{matrix} n \\ i \end{matrix} \right\} x^{n-i}, \tag{1.2}$$

where $\left\{ \begin{matrix} n \\ i \end{matrix} \right\}$ is defined as before.

Torreto and Fuchs [21] considered the general second order recurrence relation

$$y_{n+2} = g y_{n+1} - h y_n, \qquad h \neq 0. \tag{1.3}$$

Let α and β be the roots of the auxiliary polynomial $f(x) = x^2 - gx + h$ of (1.3). Let U_n and V_n be two solutions of (1.3) defined by

$$U_n = \begin{cases} \frac{\alpha^n - \beta^n}{\alpha - \beta} & \text{if } \alpha \neq \beta \\ n\alpha^{n-1} & \text{if } \alpha = \beta \end{cases} \qquad \text{and} \qquad V_n = \alpha^n + \beta^n.$$

In [10, pp. 42–45], Jarden showed that the kth order recurrence relation

$$\sum_{j=0}^{k} (-1)^j \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_U h^{\frac{j(j+1)}{2}} z_{n+k-j} = 0$$

holds for the product z_n of the nth terms of $k-1$ sequences satisfying (1.3), where

$$\left\{ \begin{matrix} k \\ j \end{matrix} \right\}_U = \frac{U_k U_{k-1} \cdots U_{k-j+1}}{U_1 U_2 \cdots U_j}, \qquad \left\{ \begin{matrix} k \\ 0 \end{matrix} \right\}_U = 1.$$

In [21], the authors established some identities involving the $\left\{ \begin{matrix} k \\ j \end{matrix} \right\}_U$, one of which is the formula:

$$\sum_{j=0}^{k} (-1)^j \left\{ \begin{matrix} k \\ j \end{matrix} \right\}_U h^{\frac{j(j+1)}{2}} U_{a_1+k-j} U_{a_2+k-j} \cdots U_{a_k+k-j} y_{n+k-j} = U_1 U_2 \cdots U_k y_{n+a_1+\cdots+a_k+k(k+1)/2},$$

where y_n and U_n satisfy (1.3) and n and the a's are any integers.

Melham [16] derived families of identities between sums of powers of the Fibonacci and Lucas numbers. In his work, while deriving these identities, he conjectured a complex identity between the Fibonacci and Lucas numbers. We recall this conjecture:

(a) Let $k, m, n \in \mathbb{Z}$ with $m > 0$, show that

$$\sum_{j=0}^{m-1} \frac{F_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m+3)}{2}} \frac{F_{n-mk}^{m+1}}{\prod_{j=1}^{m} F_{(m+1)k+j}} = F_{(m+1)(n+\frac{m}{2})}. \tag{1.4}$$

(b) The Lucas counterpart of (a) is given by

$$\sum_{j=0}^{m-1} \frac{L_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-1)} F_{(m+1)k+m-j}} + (-1)^{\frac{m(m+3)}{2}} \frac{L_{n-mk}^{m+1}}{\prod_{j=1}^{m} F_{(m+1)k+j}}$$

$$= \begin{cases} 5^{\frac{m+1}{2}} F_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is odd,} \\ 5^{\frac{m}{2}} L_{(m+1)(n+\frac{m}{2})} & \text{if } m \text{ is even,} \end{cases}$$

where $(F_n)_{(m)}$ is the "falling" factorial, which begins at F_n for $n \neq 0$, and is the product of m Fibonacci numbers excluding F_0. For example $(F_6)_{(5)} = F_6 F_5 F_4 F_3 F_2$ and $(F_3)_{(5)} = F_3 F_2 F_1 F_{-1} F_{-2}$. For $m > 0$, define $(F_0)_{(m)} = F_{-1} F_{-2} \cdots F_{-m}$ and $(F_0)_{(0)} = 1$.

In this paper, we first rearrange the conjecture of Melham by using Fibonomial coefficients instead of the "falling Fibonacci factorial". After this, we give a solution of the conjecture by translating it into a q-expression; we are left with the evaluation of a certain sum. This is achieved using contour integration.

2. The Conjecture in Terms of Fibonomials

In this section, before solving the conjecture, we will need to rewrite it via the Fibonomials. One can obtain the following version of the conjecture (a) by considering the definitions of the falling Fibonacci factorial and the Fibonomials coefficients; the following equation is equivalent to (1.4).

$$\sum_{j=0}^{m-1} \frac{F_{n+k+m-j}^{m+1}}{(F_{m-j-1})_{(m-j-1)}(F_0)_j F_{(m+1)k+m-j}} \frac{(F_{(m+1)k+m})_{(m+1)k+m}}{(F_{(m+1)k})_{(m+1)k}}$$

$$+ (-1)^{\frac{m(m+3)}{2}} F_{n-mk}^{m+1} = \left(\prod_{j=1}^{m} F_{(m+1)k+j} \right) F_{(m+1)(n+\frac{m}{2})}.$$

Notice that

$$(F_0)_j = (F_j)_j (-1)^{[\# \text{ of even positive integers} \leq j]} = (F_j)_j (-1)^{\frac{j(j+3)}{2}}.$$

From this the following conjecture is straight-forward.

Conjecture 1. *For any integers m, n and k,*

$$\sum_{j=0}^{m-1} (-1)^{\frac{j(j+3)}{2}} \left\{ \begin{array}{c} (m+1)k+m \\ j \end{array} \right\} \left\{ \begin{array}{c} (m+1)k+m-j-1 \\ m-j-1 \end{array} \right\} F_{n+k+m-j}^{m+1}$$

$$+ (-1)^{\frac{m(m+3)}{2}} F_{n-mk}^{m+1} = \left(\prod_{j=1}^{m} F_{(m+1)k+j} \right) F_{(m+1)(n+\frac{m}{2})}.$$

Proof. By taking $q = \beta/\alpha$, the claimed equality is reduced to the following form:

$$(1 - q^{(m+1)k+m}) \begin{bmatrix} (m+1)k+m-1 \\ m-1 \end{bmatrix}_q \sum_{j=0}^{m-1} (-1)^j q^{j(j+1)/2} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \frac{(1 - q^{n+k+m-j})^{m+1}}{1 - q^{(m+1)k+m-j}}$$

$$= \left(1 - q^{\frac{(m+1)(2n+m)}{2}}\right) \frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}} - (-1)^m q^{\frac{m(m+1)(2k+1)}{2}} (1 - q^{n-mk})^{m+1} \qquad (2.1)$$

where $\begin{bmatrix} m \\ j \end{bmatrix}_q$ stands for the Gaussian q-binomial coefficient:

$$\begin{bmatrix} m \\ j \end{bmatrix}_q := \frac{(q;q)_m}{(q;q)_j (q;q)_{m-j}}$$

with

$$(z;q)_n := (1-z)(1-zq)\cdots(1-zq^{n-1}).$$

Define

$$S := \sum_{j=1}^{m-1} (-1)^j q^{j(j+1)/2} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \frac{(1 - q^{n+k+m-j})^{m+1}}{1 - q^{(m+1)k+m-j}}.$$

By contour integration (for similar examples, see [18]),

$$I_R = \frac{1}{2\pi i} \oint \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1 - zq^{n+k+m})^{m+1}}{1 - zq^{(m+1)k+m}} \, dz.$$

The integration is over a large circle with radius R. We evaluate the integral by residues. The poles at $q^{-1}, \ldots, q^{-(m-1)}$ lead to our sum, but there are other poles: at $z = 0$, at $z = 1$, and at $z = q^{-(m+1)k-m}$. Then we get

$$I_R = -S + \operatorname{Res}_{z=0} \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1 - zq^{n+k+m})^{m+1}}{1 - zq^{(m+1)k+m}}$$

$$+ \operatorname{Res}_{z=1} \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1 - zq^{n+k+m})^{m+1}}{1 - zq^{(m+1)k+m}}$$

$$+ \operatorname{Res}_{z=q^{-(m+1)k-m}} \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1 - zq^{n+k+m})^{m+1}}{1 - zq^{(m+1)k+m}}.$$

For the reader's convenience, we compute one of these residues in a separate computation:

$$\operatorname{Res}_{z=1} \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1 - zq^{n+k+m})^{m+1}}{1 - zq^{(m+1)k+m}}$$

$$= \lim_{z \to 1} (z-1) \frac{(q;q)_{m-1}}{(z;q)_m} \frac{1}{z} \frac{(1 - zq^{n+k+m})^{m+1}}{1 - zq^{(m+1)k+m}}$$

$$= -\frac{(q;q)_{m-1}}{(zq;q)_{m-1}} \frac{1}{z} \frac{(1 - zq^{n+k+m})^{m+1}}{1 - zq^{(m+1)k+m}} \bigg|_{z=1}$$

$$= -\frac{(1 - q^{n+k+m})^{m+1}}{1 - q^{(m+1)k+m}}.$$

Note that as $|z|$ gets large,

$$\frac{(q;q)_{m-1}}{(z;q)_m}\frac{1}{z}\frac{(1-zq^{n+k+m})^{m+1}}{1-zq^{(m+1)k+m}} \sim \frac{1}{z}\frac{(q;q)_{m-1}}{q^{\binom{m}{2}}}\frac{q^{(n+k+m)(m+1)}}{q^{(m+1)k+m}}$$

$$= \frac{1}{z}(q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}}.$$

Consequently, as $R \to \infty$,

$$I_R \to (q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}}.$$

Thus,

$(q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}}$

$$= -S + (q;q)_{m-1} - \frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}} + [(z-q^{-(m+1)k-m})^{-1}]\frac{(q;q)_{m-1}}{(z;q)_m}\frac{1}{z}\frac{(1-zq^{n+k+m})^{m+1}}{1-zq^{(m+1)k+m}}.$$

So we obtain

$$S = -(q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}} + (q;q)_{m-1} - \frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}}$$

$$- \frac{(q;q)_{m-1}}{(q^{-(m+1)k-m};q)_m}(1-q^{n-mk})^{m+1}$$

$$= -(q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}} + (q;q)_{m-1} - \frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}}$$

$$- (-1)^m\frac{(q;q)_{m-1}(q;q)_{(m+1)k}q^{m(m+1)k+\frac{m(m+1)}{2}}}{(q;q)_{(m+1)k+m}}(1-q^{n-mk})^{m+1}.$$

Now we must prove that

$$(1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\m-1\end{bmatrix}_q\frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}}$$

$$- (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\m-1\end{bmatrix}_q(q;q)_{m-1}q^{n(m+1)+\frac{m(m+1)}{2}}$$

$$+ (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\m-1\end{bmatrix}_q(q;q)_{m-1}$$

$$- (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\m-1\end{bmatrix}_q\frac{(1-q^{n+k+m})^{m+1}}{1-q^{(m+1)k+m}}$$

$$- (1-q^{(m+1)k+m})\begin{bmatrix}(m+1)k+m-1\\m-1\end{bmatrix}_q$$

$$\times (-1)^m\frac{(q;q)_{m-1}(q;q)_{(m+1)k}q^{m(m+1)k+\frac{m(m+1)}{2}}}{(q;q)_{(m+1)k+m}}(1-q^{n-mk})^{m+1}$$

$$= \left(1-q^{\frac{(m+1)(2n+m)}{2}}\right)\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}} - (-1)^mq^{\frac{m(m+1)(2k+1)}{2}}(1-q^{n-mk})^{m+1}.$$

We gradually simplify the equation that must be proved:

$$- (1 - q^{(m+1)k+m}) \begin{bmatrix} (m+1)k + m - 1 \\ m-1 \end{bmatrix}_q (q;q)_{m-1} q^{n(m+1)+\frac{m(m+1)}{2}}$$

$$+ (1 - q^{(m+1)k+m}) \begin{bmatrix} (m+1)k + m - 1 \\ m-1 \end{bmatrix}_q (q;q)_{m-1}$$

$$- (1 - q^{(m+1)k+m}) \begin{bmatrix} (m+1)k + m - 1 \\ m-1 \end{bmatrix}_q$$

$$\times (-1)^m \frac{(q;q)_{m-1}(q;q)_{(m+1)k} q^{m(m+1)k+\frac{m(m+1)}{2}}}{(q;q)_{(m+1)k+m}} (1 - q^{n-mk})^{m+1}$$

$$= (1 - q^{\frac{(m+1)(2n+m)}{2}}) \frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}} - (-1)^m q^{\frac{m(m+1)(2k+1)}{2}} (1 - q^{n-mk})^{m+1},$$

or

$$- (1 - q^{(m+1)k+m}) \begin{bmatrix} (m+1)k + m - 1 \\ m-1 \end{bmatrix}_q (q;q)_{m-1} q^{n(m+1)+\frac{m(m+1)}{2}}$$

$$+ (1 - q^{(m+1)k+m}) \begin{bmatrix} (m+1)k + m - 1 \\ m-1 \end{bmatrix}_q (q;q)_{m-1}$$

$$= (1 - q^{\frac{(m+1)(2n+m)}{2}}) \frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}},$$

or

$$(1 - q^{(m+1)k+m}) \begin{bmatrix} (m+1)k + m - 1 \\ m-1 \end{bmatrix}_q (q;q)_{m-1} \left(1 - q^{n(m+1)+\frac{m(m+1)}{2}} \right)$$

$$= \left(1 - q^{\frac{(m+1)(2n+m)}{2}} \right) \frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}}.$$

Consequently we are left to prove that

$$(1 - q^{(m+1)k+m}) \begin{bmatrix} (m+1)k + m - 1 \\ m-1 \end{bmatrix}_q (q;q)_{m-1} = \frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}}.$$

Since this is true, the proof is complete. $\qquad\qquad\square$

Note that this proof establishes (2.1) for all values of q, not just for $q = \beta/\alpha$.

Similarly the Lucas counterpart of the conjecture is rewritten in terms of the Fibonomials as follows: for odd m,

$$\sum_{j=0}^{m-1} (-1)^{\frac{j(j+3)}{2}} \left\{ \begin{matrix} (m+1)k + m \\ j \end{matrix} \right\} \left\{ \begin{matrix} (m+1)k + m - j - 1 \\ m-j-1 \end{matrix} \right\} L^{m+1}_{n+k+m-j}$$

$$+ (-1)^{\frac{m(m+3)}{2}} L^{m+1}_{n-mk} = 5^{\frac{m+1}{2}} \left(\prod_{j=1}^{m} F_{(m+1)k+j} \right) F_{(m+1)(n+\frac{m}{2})},$$

and for even m,

$$\sum_{j=0}^{m-1}(-1)^{\frac{j(j+3)}{2}}\left\{\begin{matrix}(m+1)k+m\\j\end{matrix}\right\}\left\{\begin{matrix}(m+1)k+m-j-1\\m-j-1\end{matrix}\right\}L_{n+k+m-j}^{m+1}$$

$$+(-1)^{\frac{m(m+3)}{2}}L_{n-mk}^{m+1}=5^{\frac{m}{2}}\left(\prod_{j=1}^{m}F_{(m+1)k+j}\right)L_{(m+1)(n+\frac{m}{2})}.$$

By taking $q=\beta/\alpha$, the above equalities are translated to the following forms:

$$(1-q^{(m+1)k+m})\left[\begin{matrix}(m+1)k+m-1\\m-1\end{matrix}\right]_q\sum_{j=0}^{m-1}(-1)^jq^{j(j+1)/2}\left[\begin{matrix}m-1\\j\end{matrix}\right]_q\frac{(1+q^{n+k+m-j})^{m+1}}{1-q^{(m+1)k+m-j}}\quad(2.2)$$

$$=\left(1-q^{\frac{(m+1)(2n+m)}{2}}\right)\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}}-(-1)^mq^{\frac{m(m+1)(2k+1)}{2}}(1+q^{n-mk})^{m+1}$$

and

$$(1-q^{(m+1)k+m})\left[\begin{matrix}(m+1)k+m-1\\m-1\end{matrix}\right]_q\sum_{j=0}^{m-1}(-1)^jq^{j(j+1)/2}\left[\begin{matrix}m-1\\j\end{matrix}\right]_q\frac{(1+q^{n+k+m-j})^{m+1}}{1-q^{(m+1)k+m-j}}\quad(2.3)$$

$$=\left(1-q^{\frac{(m+1)(2n+m)}{2}}\right)\frac{(q;q)_{(m+1)k+m}}{(q;q)_{(m+1)k}}-(-1)^mq^{\frac{m(m+1)(2k+1)}{2}}(1+q^{n-mk})^{m+1},$$

respectively. Note that (2.2) and (2.3) are the same; it is no more necessary to distinguish the parity of m.

The proof of the equality (2.2) (and (2.3)) can be done similarly to (2.1). Again, it holds for general q.

3. Acknowledgment

The constructive remarks of one referee are gratefully acknowledged.

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