POLYNOMIALS DEFINED BY A SECOND-ORDER RECURRENCE, INTERLACING ZEROS, AND GRAY CODES

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ABSTRACT. A sequence of polynomials is defined by the recurrence $P_{n+1} = (P_n + c - a)^2 - c$, with $P_0 = x - c$. Conditions are found for interlacing zeros among these polynomials, and an association between zeros and Gray codes is described. If $c = a = 2$, the polynomials are closely related to Chebyshev polynomials of the first kind.

1. Introduction

This paper presents properties of polynomials P_n defined recursively by $P_{n+1} = (P_n + c - a)^2 - c$, especially in the case that $c = a$. Following methods in [1], we prove that for many of these polynomials, the zeros of P_{n+1} interlace the set of zeros of all the polynomials $P_0, P_1, \ldots, P_n$, in a manner related to Gray codes (in Sections 3 and 7). It is remarkable that the methods in [1] apply as directly as they do to the polynomials P_n, even though the polynomials in [1] are quite different from those in the present paper. What the two families have in common is the manner in which the zeros of each new polynomial arise from the preceding polynomial by the application of a "lower function" and an "upper function". In [1], these two functions are of the form

$$(cx \pm \sqrt{c^2x^2 + 4})/2,$$

whereas in the present work, the two are of the form

$$a \pm \sqrt{x}.$$

Rahman and Schmeisser [2, pp. 196–201], discuss the subject of interlacing zeros. If $a = 2$, the polynomials P_n are shown in Section 5 to be related to Chebyshev polynomials of the first kind, of which many properties are developed in Rivlin [3] and Sloane [4].

2. The Recurrence $P_{n+1} = (P_n + c - a)^2 - c$

Suppose that a and c are nonzero complex numbers, and define polynomials $P_n = P_n(x)$ by

$$P_{n+1} = (P_n + c - a)^2 - c, \tag{1}$$

where $P_0 = x - c$. For $n \geq 1$, the set $\mathcal{S}_n$ of zeros of P_n is given recursively using the functions

$$\ell(x) = a - \sqrt{x} \quad \text{and} \quad u(x) = a + \sqrt{x}, \tag{2}$$

starting with the zero $r_{01} = c$ of P_0, so that $\mathcal{S}_0 = \{r_{01}\}$. Let $r_{11} = \ell(r_{01})$ and $r_{12} = u(r_{01})$. Then

$$(r_{1j} - a)^2 = c$$

for $j = 1, 2$, so that $\mathcal{S}_1 = \{r_{11}, r_{12}\}$. For any r in $\mathcal{S}_1$, let $\rho_1 = \ell(r)$ and $\rho_2 = u(r)$. Then

$$((\rho_j - a)^2 - a)^2 = c,$$

so that for $j = 1, 2$, the number ρ_j is a zero of the polynomial

$$P_2(x) = ((x-a)^2 - a)^2 - c = P_1((x-a)^2),$$

and $\mathcal{S}_2 = \{r_{21}, r_{22}, r_{23}, r_{24}\}$, where these are the numbers ρ_j. Inductively, for $n \geq 1$, the zeros of P_n are the 2^n numbers ρ obtained by applying ℓ and u to each of the 2^{n-1} zeros of P_{n-1}, and for all x,

$$P_n(x) = P_{n-1}((x-a)^2). \tag{3}$$

Regarding the cardinality of $\mathcal{S}_n$ as 2^n, this count allows for repeated zeros. (For example, 1 is a repeated zero of P_2 when $a = c = 1$.) In Section 3, we shall present conditions for $\mathcal{S}_n$ to contain no duplicates and in Section 4, conditions for $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_n$ to contain no duplicates.

Starting with (1) written as $P_n = (P_{n-1} + c - a)^2 - c$, we have

$$P_n' = 2(P_{n-1} + c - a)P_{n-1}',$$

which leads recursively to

$$P_n' = 2^n(P_{n-1} + c - a)(P_{n-2} + c - a) \cdots (P_1 + c - a)(P_0 + c - a). \tag{4}$$

On the other hand, by (3),

$$P_n'(x) = 2(x-a)P_{n-1}'((x-a)^2). \tag{5}$$

Since P_0' is invariant of c, the same is true, by (5), for all P_n', in spite of the appearances of c in (4). To say more about this invariance, let $A_0 = x - a$, and for $n \geq 1$, define A_n by taking $c = a$ in (1), so that $A_{n+1} = A_n^2 - a$. Then, we shall quickly prove,

$$A_n - P_n = c - a. \tag{6}$$

First note that (6) holds for $n = 0$, and assume for arbitrary $k \geq 0$ that $A_k - P_k = c - a$. Then

$$\begin{aligned} A_{k+1} - P_{k+1} &= A_k^2 - a - [(P_k + c - a)^2 - c] \\ &= (A_k - P_k - c + a)(A_k + P_k + c - a) + c - a \\ &= c - a. \end{aligned}$$

To summarize, for given a and any two choices of c, for each n the two polynomials P_n differ by a constant.

Suppose that $c = a$, that $n \geq 0$, and that r is any zero of P_n. Then

$$\begin{aligned} P_{n+1}(r) &= P_n^2(r) - a = -a; \\ P_{n+2}(r) &= P_{n+1}^2(r) - a = a^2 - a = P_0(a^2); \end{aligned}$$

and inductively,

$$P_{n+k}(r) = P_{k-2}(a^2).$$

That is, we have a very short formula for $P_m(r)$ if $m \geq n$. On the other hand, suppose that $0 \leq m < n$, and let r_k denote the greatest zero of P_k. Then

$$P_m(r_n) = r_{n-m} - a.$$

In general, if r_{nj} is coded as a word $a_1 a_2 \cdots a_n$ over the alphabet $\{\ell, u\}$, as in (9)–(11), and if $r_1', r_2', \ldots, r_n'$ are the zeros coded by the words $a_1, a_1 a_2, \ldots, a_1 a_2 \cdots a_n$, respectively, then

$$P_m(r_{nj}) = r_{n-m}' - a.$$

3. The Conditions $a^2 > c > 0$ and $a \geq 2$

In this section, we are interested in conditions under which the zeros of P_n are real and distinct. These conditions lead to the next section on interlaced zeros.

Theorem 1. *Suppose that $n \geq 0$. Then the conditions*

$$a^2 > c > 0 \quad \textit{and} \quad a \geq 2$$

hold if and only if the zeros of P_n are distinct positive real numbers.

Proof. Given the hypothesis, let ℓ and u be as in (2), and define

$$u_0 = c, \quad u_1 = u(u_0), \quad \ldots, \quad u_n = u(u_{n-1}).$$

Clearly u_n is the greatest zero of P_n. It is easy to prove that $u_n < 2a$ for $n \geq 1$. As a bounded strictly increasing sequence, (u_n) has a limit, L. Taking the limit of both sides of $u_{n+1} = a + \sqrt{u_n}$ gives $L = a + \sqrt{L}$, so that

$$L = (2a + 1 + \sqrt{4a + 1})/2. \tag{7}$$

Next, consider, for any $n \geq 1$, the number $\ell\,(u_n) = a - \sqrt{u_n}$. Using the hypothesis that $a \geq 2$, it is easy to prove that $\ell(u_n) > 0$, that $\ell(u_n)$ is the least zero of P_n, that $(\ell(u_n))$ is strictly decreasing, and that its limit is $a - \sqrt{L}$. (Note that $a = \sqrt{L}$ for $a = 2$, that $a > \sqrt{L}$ for $a > 2$, and that (7) holds for all $a > 0$.)

To see that the zeros of P_n are distinct, note that this is true for $n = 0$ and assume it true for arbitrary $k \geq 0$. Each zero of P_{k+1} is $\ell(r)$ or $u(r)$ for some zero r of P_k. Clearly if r_1 and r_2 are zeros of P_k then $\ell(r_1) < u(r_2)$. Moreover, if $r_1 < r_2$, then $\ell(r_1) > \ell(r_2)$ and $u(r_1) < u(r_2)$. Therefore the zeros of P_{k+1} are distinct, and by induction this is true for every P_n.

We prove the converse in cases: if $c = 0$, then the zeros of P_1 are not distinct, and if $c < 0$, the zeros of P_1 are nonreal. If $c > 0$ and $a^2 \leq c$, then by (2), the zero $\ell(c)$ of P_1 is not positive. Finally, if $a^2 > c > 0$ but $a < 2$, then

$$\inf \bigcup_{n=1}^{\infty} \mathcal{S}_n = a - \sqrt{L},$$

which is easily proved to be negative for $0 \leq a < 2$; consequently P_n has a negative zero for some n. □

The proof of Theorem 1 shows that for $n \geq 1$, the numbers in $\mathcal{S}_n$ range in strictly increasing order between $a - \sqrt{L}$ and $a + \sqrt{L}$. For $n \geq 0$, let $\mathcal{Z}_n$ denote the list of numbers in $\mathcal{S}_n$ ordered from least to greatest. For $n \geq 1$, we may speak of the lower and upper halves of $\mathcal{Z}_n$; that is, numbers $\ell(r) < a$ and numbers $u\,(r) > a$, respectively, where r ranges through $\mathcal{Z}_{n-1}$. In fact, much more can be said. Let $\ell = \ell(c)$, $u = u(c)$, $\ell u = \ell(u)$, and so on, so that each number in $\mathcal{Z}_n$ is represented as an n-letter word on the alphabet $\{\ell, u\}$; viz.,

$$\mathcal{Z}_0 = (c) = (r_{01}) \tag{8}$$
$$\mathcal{Z}_1 = (\ell, u) = (r_{11}, r_{12}) \tag{9}$$
$$\mathcal{Z}_2 = (\ell u, \ell\ell, u\ell, uu) = (r_{21}, r_{22}, r_{23}, r_{24}) \tag{10}$$
$$\mathcal{Z}_3 = (\ell uu, \ell u\ell, \ell\ell\ell, \ell\ell u, u\ell u, u\ell\ell, uu\ell, uuu), \text{ etc.} \tag{11}$$

To get from $\mathcal{Z}_n$ to $\mathcal{Z}_{n+1}$, apply the function $\ell(x) = a - \sqrt{x}$ to $\mathcal{Z}_n$ in reverse order, and then apply the function $u(x) = a + \sqrt{x}$ to $\mathcal{Z}_n$ in forward order. This makes $\mathcal{Z}_n$ an n-bit Gray code, a pattern that will be considered again in Section 7. It will be convenient to apply the operations $\cup$ and $\cap$ in the obvious manner to the *ordered* lists $\mathcal{Z}_n$.

Three simple identities characterize the relationship between the lower and upper halves of $\mathcal{Z}_n$ for $n \geq 1$. Each number r in the upper half has the form $u(\rho)$ for some ρ in $\mathcal{Z}_{n-1}$. Matching r is the number $r' = \ell(\rho)$ in the lower half. The three identities follow immediately from (2):

$$r + r' = 2a \tag{12}$$

$$r - r' = 2\sqrt{\rho} \tag{13}$$

$$rr' = a^2 - \rho. \tag{14}$$

Define

$$l_0 = c, \quad l_1 = l(l_0), \quad ..., \quad l_n = l(l_{n-1}).$$

In the proof of Theorem 1, we have already seen that the least zero of $\mathcal{Z}_n$ is $l(u_{n-1})$. It is of interest to compare the limits

$$l = \lim_{n\to\infty} l_n = (2a + 1 - \sqrt{4a+1})/2;$$

$$L = \lim_{n\to\infty} u_n = (2a + 1 + \sqrt{4a+1})/2.$$

If $a = c = m(m+1)$ for some positive integer m, then $l = m^2$ and $L = (m+1)^2$. Thus, with reference to the proof of Theorem 1,

$$\inf \bigcup_{n=0}^{\infty} \mathcal{Z}_n = m^2 - 1, \quad \sup \bigcup_{n=0}^{\infty} \mathcal{Z}_n = (m+1)^2,$$

the point being that these endpoints of the range interval for all the zeros of all the polynomials P_n are integers, and the length of the interval is $2m + 2$.

4. Interlaced Zeros

The main objective in this section is to prove that under the hypothesis of Theorem 1, for given n the set of all the zeros of the polynomials $P_0, P_1, \ldots, P_n$ interlace the zeros of P_{n+1}. We start with a definition. Suppose S and T are sets of numbers such that $|S| = |T| + 1 \geq 2$. Write the numbers in S in increasing order as $s_1, s_2, \ldots, s_m$, and those in T in increasing order as $t_1, t_2, \ldots, t_{m-1}$. Then S *interlaces* T if

$$s_1 < t_1 < s_2 < t_2 < \cdots < t_{m-1} < s_m.$$

Throughout this section, the conditions $a^2 > c > 0$ and $a \geq 2$ are assumed. By Theorem 1, for any $n \geq 0$, the 2^n numbers (or words) in $\mathcal{Z}_n$ are distinct. We now wish to extend this result by finding a condition under which the $2^{n+1} - 1$ numbers (or words) in $\mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_n$ are distinct. Suppose, to the contrary, that there is a least i such that $\mathcal{Z}_i \cap \mathcal{Z}_j \neq \varnothing$ for some $j > i$. Then there is a number $w_1 = w_1(\ell, u)$ in $\mathcal{Z}_i$ that is identical to a number $w_2 = w_2(\ell, u)$ in $\mathcal{Z}_j$. As words, w_1 and w_2 must have the same first letter, since every number (or word) with first letter ℓ is less than every number with first letter u. Then because ℓ and u are both strictly monotone, we must have $i = 0$, which is to say that $w_1 = c$ and w_2 is a zero of P_j. Since $w_2 = c$, we have $P_j(c) = 0$. Accordingly, the desired condition for $\mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_n$ to be free of duplicates is that

$$P_k(c) \neq 0 \text{ for } k = 1, 2, \ldots, n. \tag{15}$$

As an example, (15) fails for $k = 1$ in case $(c-a)^2 = c$, as when c is the number L in (7); e.g., if a has the form $m(m+1)$, then (15) fails for $c = (m+1)^2$ and also for $c = m^2$. On the other hand, if $\sqrt{c}$ is irrational, then it is inductively clear that the conditions (15) hold.

Next we consider gapsizes in $\mathcal{Z}_n$.

Lemma 1. *Suppose that $n \geq 1$ and that r_1 and r_2 are zeros of P_n satisfying $a < r_1 < r_2$. Then $u(r_2) - u(r_1) < r_2 - r_1$.*

Proof. Suppose $a < r_1 < r_2$; i.e., r_1 and r_2 are in the upper half of $\mathcal{Z}_n$. Then $\sqrt{r_2} + \sqrt{r_1} > 1$, so that

$$u(r_2) - u(r_1) = \sqrt{r_2} - \sqrt{r_1} < r_2 - r_1.$$

□

Applying Lemma 1 inductively to the lists $\mathcal{Z}_n$, we find on writing $m = 2^n$ that

$$r_{n,m} - r_{n,m-1} < r_{n,m-1} - r_{n,m-2} < \cdots < r_{n,m/2+1} - r_{n,m/2}.$$

This upper chain and the identity $r_{n,k} + r_{m,m-k+1} = 2a$ for $k = 1, 2, \ldots, m$ imply a lower chain:

$$r_{n,2} - r_{n,1} < r_{n,3} - r_{n,2} < \cdots < r_{n,m/2+1} - r_{n,m/2}.$$

The two chains of inequalities show that the longest gap in $\mathcal{Z}_n$ has length

$$r_{n,m/2+1} - r_{n,m/2} = u\ell u^{n-2} - \ell^2 u^{n-2}. \tag{16}$$

Note that

$$u\ell(L) = u(a - \sqrt{L}) = a + \sqrt{a - \sqrt{L}},$$
$$\ell^2(L) = \ell(a - \sqrt{L}) = a - \sqrt{a - \sqrt{L}},$$

where L is given by (7). Since $L = \lim_{n\to\infty} u^{n-2}$, we find from (16) that the limiting length of the longest gap is

$$2\sqrt{a - \sqrt{L}}. \tag{17}$$

Theorem 2. *Suppose that*

$$a^2 > c > 0 \quad \textit{and} \quad a \geq 2$$

and that $P_n(c) \neq 0$ for all $n \geq 1$. Then $\bigcup_{k=1}^{n} \mathcal{Z}_k$ interlaces $\mathcal{Z}_{n+1}$.

Proof. First, by Theorem 1, $\mathcal{Z}_n$ consists of positive real numbers for every n, and by the discussion following (15), the numbers in the union of all $\mathcal{Z}_n$ are distinct. Clearly $\mathcal{Z}_0$ interlaces $\mathcal{Z}_1$. Assume for arbitrary $n \geq 1$ that the list $\mathcal{Z} := \bigcup_{k=1}^{n-1} \mathcal{Z}_k$ interlaces $\mathcal{Z}_n$. Let $m = 2^n$. By Theorem 1,

$$|\mathcal{Z}| = m - 1, \qquad |\mathcal{Z}_n| = m, \qquad |\mathcal{Z}_{n+1}| = 2m. \tag{18}$$

Suppose that z_i and z_{i+1} are in $\mathcal{Z}_{n+1}$. As a first of three cases, assume that $c \leq z_i < z_{i+1}$. Then $z_i = u(w_1)$ and $z_{i+1} = uw_2$ for some w_1 and w_2 in $\mathcal{Z}_n$. By the induction hypothesis, there exists a number w in $\mathcal{Z}$ such that $w_1 < w < w_2$. Since u is strictly increasing, we have $u(w_1) < u(w) < u(w_2)$, where uw is in $\bigcup_{k=1}^{n} \mathcal{Z}_k$.

For case 2, suppose that $z_i < c < z_{i+1}$; then clearly a number in $\mathcal{Z}$ separates z_i and z_{i+1}. For case 3, assume that $z_i < z_{i+1} \leq c$. Then $z_i = \ell(w_1)$ and $z_{i+1} = \ell(w_2)$, where w_1 and w_2 are numbers in $\mathcal{Z}_n$ satisfying $w_2 < w_1$. By the induction hypothesis, there exists w in $\mathcal{Z}$ such that $w_2 < w < w_1$, and since ℓ is strictly decreasing, we have $\ell(w_1) < u(w) < \ell(w_2)$. Thus, in all three cases, there is at least one number z in $\mathcal{Z}$ between each pair of numbers z_i and z_{i+1} in $\mathcal{Z}_{n+1}$. The cardinalities in (18) thus imply that there is exactly one such z. Moreover, $\ell u^n < z < u^{n+1}$ for all z in $\mathcal{Z}$. Therefore, $\bigcup_{k=1}^{n} \mathcal{Z}_k$ interlaces $\mathcal{Z}_{n+1}$. □

A second proof of Theorem 2 follows. Putting $c = a$ in (4) and adjusting subscripts give $P'_{n+1} = 2^n P_n P_{n-1} \cdots P_1 P_0$, a polynomial whose $2^{n+1} - 1$ zeros comprise $\bigcup_{k=1}^{n} \mathcal{Z}_k$, while the zeros of P_{n+1} comprise $\mathcal{Z}_{n+1}$. Between each neighboring pair of zeros of P_{n+1} occurs a local extreme of P_{n+1}, so that there must be $2^{n+1} - 1$ such extremes. Since P'_{n+1} has exactly $2^{n+1} - 1$ zeros and these are the x-coordinates of the local extremes of P_{n+1}, they are interlaced by the zeros of P_n. If $c \neq a$ (but still $a^2 > c > 0$), then by (6), $P'_{n+1} = A'_{n+1}$, so that in this case, too, the interlacing holds.

5. The Case $a = c = 2$

Throughout this section, assume that $a = c = 2$. The first four polynomials P_n are then

$$\begin{aligned}
P_0 &= x - 2\\
P_1 &= x^2 - 4x + 2\\
P_2 &= x^4 - 8x^3 + 20x^2 - 16x + 2\\
P_3 &= x^8 - 16x^7 + 104x^6 - 352x^5 + 660x^4 - 672x^3 + 336x - 64x + 2.
\end{aligned}$$

Further coefficients are given [4] as A158982. We shall see that the sequence P_n is closely related to a subsequence of the sequence T_n of Chebyshev polynomials of the first kind. These classical polynomials are defined for $n = 0, 1, 2, \ldots$ by

$$T_n(x) = \cos(n \arccos x). \tag{19}$$

The first five are

$$\begin{aligned}
T_1 &= 1\\
T_2 &= x\\
T_3 &= 2x^2 - 1\\
T_4 &= 4x^3 - 3x\\
T_5 &= 8x^4 - 8x^2 + 1.
\end{aligned}$$

The definition (19) implies that the sequence T_n is given recursively by

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x),$$

and, that among dozens [3] of well-known of identities,

$$\begin{aligned}
T_n(x) &= \frac{1}{2}\left[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right];\\
T_m \circ T_n &= T_{mn};\\
T_m \cdot T_n &= \frac{1}{2}\left(T_{m+n} + T_{|m-n|}\right);\\
(1 - x^2)T''_n - xT'_n + n^2 T_n &= 0.
\end{aligned}$$

The polynomials P_n are related to Chebyshev polynomials by the identity

$$P_n(x) = 2T_{2^{n+1}}(\sqrt{x}/2),$$

so that properties of the Chebyshev polynomials imply properties of the polynomials P_n. The connection between the two families of polynomials is indicated in [4] at A084534, which is described as the "unsigned version of the coefficient table for scaled Chebyshev $T(2 * n, x)$

polynomials." For example, among the rows of that table are $(1, 4, 2)$ and $(1, 8, 20, 16, 2)$, compared with coefficients $(1, -4, 2)$ and $(1, -8, 20, -16, 2)$ for P_1 and P_2.

Next, we determine the zeros of P_n from those of T_n.

$$\begin{aligned} T_n(x) &= 2^{n-1} \prod_{k=1}^{n} (x - \cos \frac{(2k-1)\pi}{2n}); \\ P_n(x) &= 2T_{2^{n+1}}(\frac{\sqrt{x}}{2}) \\ &= 2^n \prod_{k=1}^{2^{n+1}} (\frac{\sqrt{x}}{2} - \cos \frac{(2k-1)\pi}{2^{n+2}}); \\ P_n(x^2) &= 2^n \prod_{k=1}^{2^{n+1}} \frac{1}{2}(x - 2\cos \frac{(2k-1)\pi}{2^{n+2}}). \end{aligned}$$

Now applying (3),

$$P_{n+1}(x+2) = \frac{1}{2} \prod_{k=1}^{2^{n+1}} (x - 2\cos \frac{(2k-1)\pi}{2^{n+2}}).$$

Let $t = x + 2$:

$$P_{n+1}(t) = \frac{1}{2} \prod_{k=1}^{2^{n+1}} (t - 2 - 2\cos \frac{(2k-1)\pi}{2^{n+2}}),$$

so that the zeros of P_n are

$$2 + 2\cos \frac{(2k-1)\pi}{2^{n+1}},$$

for $k = 1, 2, \ldots, 2^n$. The least of these zeros is

$$2 - 2\cos \frac{\pi}{2^{n+1}}. \tag{20}$$

With $a = 2$ in (7), we already know that the limit of the least zero $\ell(u_n)$ is 0; a stronger result, obtained from (20), is that $\sum_{n=0}^{\infty} \ell(u_n) < \infty$. Specifically,

$$2\sum_{n=0}^{\infty} \left(1 - \cos \frac{\pi}{2^{n+1}}\right) = 2.78929960425033\ldots.$$

We return now to the supremum (17) for the gapsize between adjacent numbers in $\bigcup_{n=1}^{\infty} \mathcal{Z}_n$. It is easy to check that if $a > 0$ and $2\sqrt{a - \sqrt{L}} = 0$, then $a = 2$. Accordingly, $\bigcup_{n=1}^{\infty} \mathcal{Z}_n$ is dense in the interval $[a - \sqrt{L}, a + \sqrt{L}]$ if and only if $a = 2$.

6. The Case $a = c = 1$

Throughout this section, assume that $a = c = 1$ so that the first four polynomials P_n are

$$\begin{aligned} P_0 &= x - 1 \\ P_1 &= x^2 - 2x \\ P_2 &= x^4 - 4x^3 + 4x^2 - 1 \\ P_3 &= x^8 - 8x^7 + 24x^6 - 32x^5 + 14x^4 + 8x^3 - 8x^2, \end{aligned}$$

as in [4] at A158984. By (1),

$$\begin{aligned} P_n &= (P_{n-1} - 1)(P_{n-1} + 1) \\ &= (P_{n-1} - 1)P_{n-2}^2. \end{aligned} \tag{21}$$

Obviously, P_n is highly composite for large n; its factor

$$Q_n := P_{n-1} - 1$$

is of some interest. The first three of these polynomials are

$$\begin{aligned} Q_1 &= x - 2 \\ Q_2 &= x^2 - 2x - 1 \\ Q_3 &= x^4 - 4x^3 + 4x^2 - 2, \end{aligned}$$

as in [4] at A158986. Iterating (21) gives

$$P_n = \begin{cases} (x-1)^{2^k} Q_n Q_{n-2}^2 Q_{n-4}^4 \cdots Q_2^{2^{k-1}} & \text{if } n = 2k \text{ is even} \\ \\ (x^2-2x)^{2^k} Q_n Q_{n-2}^2 Q_{n-4}^4 \cdots Q_3^{2^{k-1}} & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

A recurrence for Q_n is easily found:

$$\begin{aligned} Q_n &= P_n^2 - 2 \\ &= (Q_{n-1} + 1)^2 - 2 \\ &= (Q_{n-1} + 1 - \sqrt{2})(Q_{n-1} + 1 + \sqrt{2}). \end{aligned}$$

The zeros of Q_2 are $\ell(2) = 1 - \sqrt{2} < 0$ and $u(2) = 1 + \sqrt{2} > 0$, so that the 4 zeros of Q_3 are given by

$$\ell(u(2)) < 0; \qquad \ell(\ell(2)),\ \text{nonreal}; \qquad u(\ell(2)),\ \text{nonreal}; \qquad u(u(2)) > 0.$$

For arbitrary $k \geq 3$, assume as an induction hypothesis that, in regard to the 2^k zeros of Q_k, one is negative, one is positive, and all the others are nonreal. Then the zeros of Q_{k+1}, obtained by applying the functions $\ell(x)$ and $u(x)$ in (2) to the zeros of Q_k, have the same distribution: one negative, one positive, and all others nonreal. Therefore, this distribution holds for every $n \geq 3$.

The increasing sequence of positive zeros, $(1 + \sqrt{2}, 1 + \sqrt{1 + \sqrt{2}}, \ldots)$ is easily seen to have limit $1 + \tau$, where $\tau = (1 + \sqrt{5})/2$, the golden ratio. The decreasing sequence of negative zeros, $(1 - \sqrt{2}, 1 - \sqrt{1 + \sqrt{2}}, \ldots)$, has limit $1 - \tau$.

7. Concluding Remarks

Consider the zeros of P_3 in the case that $c = a \geq 2$:

$$\begin{array}{ll}
r_{31} = l(r_{24}) = a - \sqrt{a + \sqrt{a + \sqrt{a}}} & - + + \\
r_{32} = l(r_{23}) = a - \sqrt{a + \sqrt{a - \sqrt{a}}} & - + - \\
r_{33} = l(r_{22}) = a - \sqrt{a - \sqrt{a - \sqrt{a}}} & - - - \\
r_{34} = l(r_{21}) = a - \sqrt{a - \sqrt{a + \sqrt{a}}} & - - + \\
r_{35} = u(r_{21}) = a + \sqrt{a - \sqrt{a + \sqrt{a}}} & + - + \\
r_{36} = u(r_{22}) = a + \sqrt{a - \sqrt{a - \sqrt{a}}} & + - - \\
r_{37} = u(r_{23}) = a + \sqrt{a + \sqrt{a - \sqrt{a}}} & + + - \\
r_{38} = u(r_{24}) = a + \sqrt{a + \sqrt{a + \sqrt{a}}} & + + +.
\end{array}$$

As an observation presaged in Section 3, the eight sign-triples, beginning with $- + +$ and ending with $+ + +$, constitute a Gray code. That is, as we pass from each triple to the next, there is exactly one change in sign, and this is also true as we pass from the final $+ + +$ to the original $- + +$. Remarkably, for any $a \geq 2$, we have, as a match for the Gray code ordering, the monotonic ordering given by

$$r_{31} < r_{32} < r_{33} < r_{34} < r_{35} < r_{36} < r_{37} < r_{38},$$

and similarly for all $n \geq 0$. If we regard a as a variable, then the expressions r_{ij} represent *functions* of a that have the interlacing properties proved in Section 4 for numerical values. These observations about Gray codes and interlacing functions apply also to the polynomials discussed in [1].

With the ordered lists (8)–(11) in mind, it is natural to consider the position of each number r_{ij} within the successive ordered-union lists

$$\mathcal{Z}_0, \qquad \mathcal{Z}_0 \cup \mathcal{Z}_1, \qquad \mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \mathcal{Z}_2, \qquad \mathcal{Z}_0 \cup \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3, \ldots .$$

For $\mathcal{Z}_0$ we write 1, as r_{01} is the only r_{ij}. For $\mathcal{Z}_0 \cup \mathcal{Z}_1$ we write 2 1 3 for the positions of r_{11}, r_{01}, r_{12}, and so on. The first four rows of the resulting array, A131987, named "Representation of a dense para-sequence," are

							1							
						2	1	3						
				4	2	5	1	6	3	7				
8	4	9	2	10	5	11	1	12	6	13	3	14	7	15

The interlacing property is especially easily seen here; e.g., the odd-numbered terms $8, 9, 10, \ldots, 15$ in row 4 interlace the terms $4, 2, 5, 1, 6, 3, 7$ comprising row 3.

Next, we note that if $c = a$ is a nonzero integer, then (1) clearly implies $P_n(x) \equiv x^n \bmod a$ for $n = 0, 1, 2, \ldots$, since every coefficient in P_n, except the coefficient of x^n, is an integer multiple of a.

Finally, for various choices of a, c, and k, the integer sequence $P_n(k)$ is of interest. For example, when $c = a = -1$, the sequence $P_n(1)$ is essentially A003095 in [4], associated with the number of binary trees of height less than n. In connection with $a = -2$, there are (with reference to the encyclopedia [4]) the Lucas-Lehmer sequence A003010 and related sequences A001566, A003487, A003423, A072191, and A102847.

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