SEQUENCES $\{H_n\}$ FOR WHICH H_{n+1}/H_n APPROACHES AN IRRATIONAL NUMBER

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ABSTRACT. We study the sequence $\{H_n\}$ for which H_{n+1}/H_n approaches an irrational number. This result includes a special case obtained by Gatta and D'Amico. We also generalize it to the case of the sth order recurrence relation.

1. INTRODUCTION

It is well-known [3, Corollary 8.5] that the ratio of Fibonacci numbers F_n tends to the Golden number:

$$\frac{F_{n+1}}{F_n} \to \frac{1+\sqrt{5}}{2} \quad (n \to \infty)\,.$$

Gatta and D'Amico [1] construct the sequence $\{H_n\}$, which ratio also tends to the Golden number. In this paper we generalize their result and consider much more general cases.

Theorem 1.1. *Let c_1, c_2 and k be fixed real numbers with $c_1^2+4c_2>0$ and $c_1+c_2 \neq 1$. Let χ_1 and $\chi_2 (\neq 1)$ be the roots of the equation $x^2-c_1x-c_2=0$, namely,*

$$\chi_1 = \frac{c_1+\sqrt{c_1^2+4c_2}}{2} \quad \text{and} \quad \chi_2 = \frac{c_1-\sqrt{c_1^2+4c_2}}{2}\,.$$

Let H_1, H_2, and H_3 be arbitrary real numbers with $H_3 = c_1H_2+c_2H_1+k$. Then, the recurrence relations

$$H_n = c_1H_{n-1}+c_2H_{n-2}+k \quad (n \geq 4) \tag{1.1}$$

and

$$H_n = (c_1+1)H_{n-1}+(-c_1+c_2)H_{n-2}-c_2H_{n-3} \quad (n \geq 4) \tag{1.2}$$

yield the same sequence $\{H_n\}$, satisfying

$$H_n = \alpha_nH_2+\beta_nH_1+\gamma_nk \quad (n \geq 1) \tag{1.3}$$

and

$$H_n = r_nH_3+s_nH_2+t_nH_1 \quad (n \geq 1)\,, \tag{1.4}$$

where

$$\alpha_n = \frac{\chi_1^{n-1}-\chi_2^{n-1}}{\chi_1-\chi_2}, \quad \beta_n = \frac{c_2(\chi_1^{n-2}-\chi_2^{n-2})}{\chi_1-\chi_2},$$
$$\gamma_n = \frac{1}{\chi_2-1}\left(\frac{\chi_1^{n-1}-\chi_2^{n-1}}{\chi_1-\chi_2} - \frac{\chi_1^{n-1}-1}{\chi_1-1}\right) \quad (n \geq 1)$$

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and

$$r_n = \frac{1}{\chi_1 - \chi_2}\left(\frac{\chi_1^{n-1} - \chi_1}{\chi_1 - 1} - \frac{\chi_2^{n-1} - \chi_2}{\chi_2 - 1}\right),$$
$$s_n = 1 - \frac{1}{\chi_1 - \chi_2}\left(\frac{(\chi_1 - c_2)(\chi_1^{n-2} - 1)}{\chi_1 - 1} - \frac{(\chi_2 - c_2)(\chi_2^{n-2} - 1)}{\chi_2 - 1}\right),$$
$$t_n = -\frac{c_2}{\chi_1 - \chi_2}\left(\frac{\chi_1^{n-2} - 1}{\chi_1 - 1} - \frac{\chi_2^{n-2} - 1}{\chi_2 - 1}\right) \quad (n \ge 1)$$

with $r_n + s_n + t_n = 1$ *for all* $n \ge 1$. *Moreover,*

$$\frac{H_{n+1}}{H_n} \to \chi_1 \quad (n \to \infty).$$

Remark. *If* $c_1 = c_2 = 1$, *then this result reduces to Theorem 2.1 and Corollary 2.2 in* [1].

Example 1.2. *Set* $c_1 = 5/6$ *and* $c_2 = 37/11$. *If* $H_0 = 2/5$, $H_1 = 4/7$, *and* $H_2 = 2/9$, *then*

$$\frac{H_2}{H_1} = \frac{7}{18} = 0.388888 \qquad \frac{H_3}{H_2} = \frac{5279}{2310} = 2.285281385$$
$$\frac{H_4}{H_3} = -\frac{26741}{31674} = -0.844257119 \qquad \frac{H_5}{H_4} = \frac{1023023}{1764906} = 0.57964730$$
$$\frac{H_6}{H_5} = \frac{80237503}{6138138} = 13.071961399 \qquad \frac{H_7}{H_6} = \frac{8382952117}{5295675198} = 1.58298079$$
$$\frac{H_8}{H_7} = \frac{164434451477}{50297712702} = 3.2692232438 \qquad \frac{H_9}{H_8} = \frac{21242447286215}{10852673797482} = 1.957346888$$
$$\frac{H_{10}}{H_9} = \frac{66286737441851}{25490936743458} = 2.6004041400 \qquad \frac{H_{11}}{H_{10}} = \frac{46932646834861261}{21874623355810830} = 2.1455293685$$

....

$$\frac{H_{23}}{H_{22}} = 2.29668165729 \qquad \frac{H_{24}}{H_{23}} = 2.29789732722$$
$$\frac{H_{25}}{H_{24}} = 2.29712230004 \qquad \frac{H_{26}}{H_{25}} = 2.29761607245$$
$$\frac{H_{27}}{H_{26}} = 2.29730134694 \qquad \frac{H_{28}}{H_{27}} = 2.29750188932$$
$$\frac{H_{29}}{H_{28}} = 2.29737407831$$

On the other hand, if $c_1 = 5/6$ *and* $c_2 = 37/11$, *then*

$$\frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2} = 2.29742382131143226702631437357.$$

Proof of Theorem 1.1. By the identity (1.1)

$$(c_1 + 1)H_{n-1} + (-c_1 + c_2)H_{n-2} - c_2 H_{n-3} = c_1 H_{n-1} + c_2 H_{n-2} + k = H_n,$$

yielding the identity (1.2) for n.

On the other hand, by the identity (1.2)

$$c_1 H_{n-1} + c_2 H_{n-2} + k = (c_1 + 1)H_{n-1} + (-c_1 + c_2)H_{n-2} - c_2 H_{n-3} = H_n,$$

yielding the identity (1.1) for n.

The identity (1.3) holds for $n = 1$ and $n = 2$ since $\alpha_1 = 0$, $\alpha_2 = 1$, $\beta_1 = 1$, $\beta_2 = 0$, $\gamma_1 = \gamma_2 = 0$. Assume that (1.3) holds for $n = l-2$ and $n = l-1$. Then, by (1.1)

$$\begin{aligned} H_l &= c_1 H_{l-1} + c_2 H_{l-2} + k \\ &= c_1(\alpha_{l-1}H_2 + \beta_{l-1}H_1 + \gamma_{l-1}k) + c_2(\alpha_{l-2}H_2 + \beta_{l-2}H_1 + \gamma_{l-2}k) + k \\ &= (c_1\alpha_{l-1} + c_2\alpha_{l-2})H_2 + (c_1\beta_{l-1} + c_2\beta_{l-2})H_1 + (c_1\gamma_{l-1} + c_2\gamma_{l-2} + 1)k. \end{aligned}$$

Thus,

$$\begin{aligned} \alpha_l &= c_1\alpha_{l-1} + c_2\alpha_{l-2}, \\ \beta_l &= c_1\beta_{l-1} + c_2\beta_{l-2}, \\ \gamma_l &= c_1\gamma_{l-1} + c_2\gamma_{l-2} + 1. \end{aligned}$$

Solving the first recurrence relation with $\alpha_1 = 0$, $\alpha_2 = 1$, we have

$$\begin{aligned} \alpha_n &= \frac{\alpha_2(\chi_1^{n-1} - \chi_2^{n-1}) + c_2\alpha_1(\chi_1^{n-2} - \chi_2^{n-2})}{\chi_1 - \chi_2} \\ &= \frac{\chi_1^{n-1} - \chi_2^{n-1}}{\chi_1 - \chi_2}. \end{aligned}$$

Solving the second recurrence relation with $\beta_1 = 1$, $\beta_2 = 0$, we have

$$\begin{aligned} \beta_n &= \frac{\beta_2(\chi_1^{n-1} - \chi_2^{n-1}) + c_2\beta_1(\chi_1^{n-2} - \chi_2^{n-2})}{\chi_1 - \chi_2} \\ &= \frac{c_2(\chi_1^{n-2} - \chi_2^{n-2})}{\chi_1 - \chi_2}. \end{aligned}$$

Solving the third recurrence relation with $\gamma_1 = \gamma_2 = 0$, we have

$$\begin{aligned} \gamma_n &= \frac{\gamma_2(\chi_1^{n-1} - \chi_2^{n-1}) + c_2\gamma_1(\chi_1^{n-2} - \chi_2^{n-2})}{\chi_1 - \chi_2} + \frac{1}{\chi_2 - 1}\left(\frac{\chi_1^{n-1} - \chi_2^{n-1}}{\chi_1 - \chi_2} - \frac{\chi_1^{n-1} - 1}{\chi_1 - 1}\right) \\ &= \frac{1}{\chi_2 - 1}\left(\frac{\chi_1^{n-1} - \chi_2^{n-1}}{\chi_1 - \chi_2} - \frac{\chi_1^{n-1} - 1}{\chi_1 - 1}\right). \end{aligned}$$

Next, the identity (1.4) holds for $n = 1, 2, 3$ as $r_1 = r_2 = 0$, $r_3 = 1$, $s_1 = 0$, $s_2 = 1$, $s_3 = 0$, $t_1 = 1$, $t_2 = t_3 = 0$. Assume that (1.4) holds for $n = l-3$, $l-2$, and $l-1$. Then by (1.2) we have

$$\begin{aligned} H_l &= (c_1 + 1)(r_{l-1}H_3 + s_{l-1}H_2 + t_{l-1}H_1) + (-c_1 + c_2)(r_{l-2}H_3 + s_{l-2}H_2 + t_{l-2}H_1) \\ &\quad - c_2(r_{l-3}H_3 + s_{l-3}H_2 + t_{l-3}H_1) \\ &= r_l H_3 + s_l H_2 + t_l H_1. \end{aligned}$$

Hence, for $n \geq 1$

$$\begin{aligned} r_n - r_{n-1} &= c_1(r_{n-1} - r_{n-2}) + c_2(r_{n-2} - r_{n-3}), \\ s_n - s_{n-1} &= c_1(s_{n-1} - s_{n-2}) + c_2(s_{n-2} - s_{n-3}), \\ t_n - t_{n-1} &= c_1(t_{n-1} - t_{n-2}) + c_2(t_{n-2} - t_{n-3}). \end{aligned}$$

Since

$$r_n - r_{n-1} = \frac{(\chi_1^{n-2} - \chi_2^{n-2})(r_3 - r_2) + c_2(\chi_1^{n-3} - \chi_2^{n-3})(r_2 - r_1)}{\chi_1 - \chi_2},$$

we have

$$r_n = \frac{1}{\chi_1 - \chi_2}\left(\frac{\chi_1^{n-1} - \chi_1}{\chi_1 - 1} - \frac{\chi_2^{n-1} - \chi_2}{\chi_2 - 1}\right).$$

Since

$$s_n - s_{n-1} = \frac{(\chi_1^{n-2} - \chi_2^{n-2})(s_3 - s_2) + c_2(\chi_1^{n-3} - \chi_2^{n-3})(s_2 - s_1)}{\chi_1 - \chi_2},$$

we have

$$s_n = 1 + \frac{1}{\chi_1 - \chi_2}\left(-\frac{\chi_1^{n-1} - \chi_1}{\chi_1 - 1} + \frac{\chi_2^{n-1} - \chi_2}{\chi_2 - 1} + \frac{c_2(\chi_1^{n-2} - 1)}{\chi_1 - 1} - \frac{c_2(\chi_2^{n-2} - 1)}{\chi_2 - 1}\right).$$

Since

$$t_n - t_{n-1} = \frac{(\chi_1^{n-2} - \chi_2^{n-2})(t_3 - t_2) + c_2(\chi_1^{n-3} - \chi_2^{n-3})(t_2 - t_1)}{\chi_1 - \chi_2},$$

we have

$$t_n = -\frac{c_2}{\chi_1 - \chi_2}\left(\frac{\chi_1^{n-2} - 1}{\chi_1 - 1} - \frac{\chi_2^{n-2} - 1}{\chi_2 - 1}\right).$$

It is clear that $r_n + s_n + t_n = 1$.

Finally,

$$\begin{aligned}
\lim_{n\to\infty}\frac{H_{n+1}}{H_n} &= \lim_{n\to\infty}\left(c_1 + \frac{c_2}{\frac{H_n}{H_{n-1}}} + \frac{k}{H_n}\right)\\
&= \lim_{n\to\infty}\left(c_1 + \frac{c_2}{c_1 + \frac{c_2}{\frac{H_{n-1}}{H_{n-2}}} + \frac{k}{H_{n-1}}}\right)\\
&= c_1 + \cfrac{c_2}{c_1 + \cfrac{c_2}{c_1 + \cfrac{c_2}{c_1 + \ddots}}}\\
&= \frac{c_1 + \sqrt{c_1^2 + 4c_2}}{2}.
\end{aligned}$$

□

2. The Sequence Derived from a General Recurrence Relation

Let $\{u_n\}$ be the sequence satisfying the sth order recurrence relation

$$u_n = u_{n-1} + u_{n-2} + \cdots + u_{n-s} \quad (n \ge s).$$

Let Φ_s be the positive real root, and $\lambda_1^{(s)}, \lambda_2^{(s)}, \ldots, \lambda_{s-1}^{(s)}$ be the other roots, of the equation $x^s - x^{s-1} - x^{s-2} - \cdots - x - 1 = 0$ [4, 5]. For example, if $s = 3$, then

$$\Phi_3 = \frac{\sqrt[3]{19 + 3\sqrt{33}} + \sqrt[3]{19 - 3\sqrt{33}} + 1}{3} = 1.839286755$$

and

$$\begin{aligned}
\lambda_1^{(3)}, \lambda_2^{(3)} &= \frac{2 - (1 \pm \sqrt{-3})\sqrt[3]{19 - 3\sqrt{33}} - (1 \mp \sqrt{-3})\sqrt[3]{19 + 3\sqrt{33}}}{6}\\
&= -0.4196433776 \pm 0.6062907292\sqrt{-1}
\end{aligned}$$

(e.g. [2, p.14] and references there).

Note that $\Phi_s \nearrow 2$ $(s \to \infty)$.

The following Lemma is fundamental.

Lemma 2.1. *If*

$$u_s - u_{s-1}\sum_{i=1}^{s-1}\lambda_i^{(s)} + u_{s-2}\sum_{1\le i<j\le s-1}\lambda_i^{(s)}\lambda_j^{(s)} - \cdots + (-1)^l u_{s-l}\sum_{1\le i_1<\cdots<i_l\le s-1}\lambda_{i_1}^{(s)}\dots\lambda_{i_l}^{(s)} + \cdots + (-1)^{s-1}u_1\lambda_1^{(s)}\lambda_2^{(s)}\dots\lambda_{s-1}^{(s)} \ne 0, \quad (2.1)$$

then

$$\frac{u_{n+1}}{u_n} \to \Phi_s \quad (n \to \infty).$$

Theorem 2.2. *Let a and b be real numbers with $|b/a| > \Phi_s$ $(a \ne 0)$. Let the sequence $\{v_n\}$ satisfy the relation*

$$av_n = (a-b)v_{n-1} + (a+b)v_{n-2} + \cdots + (a+b)v_{n-s} + bv_{n-s-1} \quad (n > s+1)$$

with arbitrary initial values $v_1, \dots, v_{s+1}$. If the condition (2.1) *holds for $u_n = av_{n+1} + bv_n$ $(n = 1, 2, \dots, s)$, then*

$$\frac{v_{n+1}}{v_n} \to \Phi_s \quad (n \to \infty).$$

If $s = 2$, $a = 1$, and $b = -1$, then this theorem is reduced to Theorem 2.1 in [1].

If $s = 3$, $a = 1$, and $b = -1$, then we have the following.

Corollary 2.3. *Let the sequence $\{v_n\}$ satisfy the relation*

$$v_n = 2v_{n-1} - v_{n-4} \quad (n > 4)$$

with arbitrary initial values v_1, v_2, v_3, v_4. If

$$v_4 - 0.160713244785839 15 v_3 - 0.2955977425220849 v_2 - 0.5436890126920759 v_1 \ne 0, \quad (2.2)$$

then

$$\frac{v_{n+1}}{v_n} \to \Phi_3 \quad (n \to \infty).$$

Notice that $T_{n+1}/T_n \to \Phi_3$ $(n \to \infty)$, where T_n are Tribonacci numbers defined by

$$T_n = T_{n-1} + T_{n-2} + T_{n-3} \quad (n \ge 4), \quad T_1 = T_2 = 1, \quad T_3 = 2.$$

The condition (2.2) means

$$(v_4 - v_3) - (v_3 - v_2)(\beta + \gamma) + (v_2 - v_1)\beta\gamma \ne 0,$$

where $\beta = \lambda_1^{(3)}$ and $\gamma = \lambda_2^{(3)}$, satisfying

$$\beta + \gamma = \frac{2 - \sqrt[3]{19 + 3\sqrt{33}} - \sqrt[3]{19 - 3\sqrt{33}}}{3} = -0.839286755214161$$

and

$$\beta\gamma = \frac{\sqrt[3]{(19+3\sqrt{33})^2} + \sqrt[3]{(19-3\sqrt{33})^2} - \sqrt[3]{19+3\sqrt{33}} - \sqrt[3]{19-3\sqrt{33}} - 3}{9}$$
$$= 0.5436890126920759\,.$$

Proof of Theorem 2.2. If we set $u_n = av_{n+1} + bv_n$, then

$$av_{n+1} + bv_n = av_n + bv_{n-1} + av_{n-1} + bv_{n-2} + \cdots + av_{n-s+1} + bv_{n-s}$$

or

$$av_{n+1} = (a-b)v_n + (a+b)v_{n-1} + \cdots + (a+b)v_{n-s+1} + bv_{n-s}\,.$$

Assume that $a \neq 0$ and $b \neq 0$.

If $v_{n+1}/v_n \to \Phi_s$ $(n \to \infty)$, then

$$\frac{u_{n+1}}{u_n} = \frac{av_{n+2} + bv_{n+1}}{av_{n+1} + bv_n} = \frac{a\frac{v_{n+2}}{v_{n+1}} + b}{a + \frac{b}{v_{n+1}/v_n}} \to \frac{a\Phi_s + b}{a + b/\Phi_s} = \Phi_s \quad (n \to \infty)\,.$$

On the other hand, assume that $u_{n+1}/u_n \to \Phi_s$. Since

$$v_{n+1} - \left(-\frac{b}{a}\right)^n v_1 = \frac{1}{a}\left(u_n + \left(-\frac{b}{a}\right)u_{n-1} + \left(-\frac{b}{a}\right)^2 u_{n-2} - \cdots + \left(-\frac{b}{a}\right)^{n-1} u_1\right),$$

we get

$$\frac{v_{n+1}}{v_n} = \frac{\left(-\frac{a}{b}\right)^n u_n + \left(-\frac{a}{b}\right)^{n-1} u_{n-1} + \left(-\frac{a}{b}\right)^{n-2} u_{n-2} + \cdots + \left(-\frac{a}{b}\right) u_1 + v_1}{\left(-\frac{a}{b}\right)^n u_{n-1} + \left(-\frac{a}{b}\right)^{n-1} u_{n-2} + \left(-\frac{a}{b}\right)^{n-2} u_{n-3} + \cdots + \left(-\frac{a}{b}\right)^2 u_1 + \left(-\frac{a}{b}\right) v_1}\,.$$

Here, notice that

$$\left|\left(-\frac{a}{b}\right)\frac{u_{n+1}}{u_n}\right| \to \Phi_s \left|\frac{a}{b}\right| \quad (n \to \infty)\,.$$

Hence, by Ratio Test, if $|a/b|\Phi_s < 1$ then the series $\sum_{n=1}^{\infty}(-a/b)^n u_n$ is absolutely convergent. If $\Phi_s|a/b| > 1$ then it is divergent. Therefore, if $\Phi_s < |b/a|$ then by $u_{n+k}/u_n \to \Phi_s^k$ $(n \to \infty)$

$$\begin{aligned}\frac{v_{n+1}}{v_n} &= \frac{\left(-\frac{a}{b}\right)^n \frac{u_n}{u_{n-1}} + \left(-\frac{a}{b}\right)^{n-1} + \left(-\frac{a}{b}\right)^{n-2}\frac{u_{n-2}}{u_{n-1}} + \cdots + \left(-\frac{a}{b}\right)\frac{u_1}{u_{n-1}} + \frac{v_1}{u_{n-1}}}{\left(-\frac{a}{b}\right)^n + \left(-\frac{a}{b}\right)^{n-1}\frac{u_{n-2}}{u_{n-1}} + \left(-\frac{a}{b}\right)^{n-2}\frac{u_{n-3}}{u_{n-1}} + \cdots + \left(-\frac{a}{b}\right)^2\frac{u_1}{u_{n-1}} + \left(-\frac{a}{b}\right)\frac{v_1}{u_{n-1}}} \\ &= \frac{\left(-\frac{a}{b}\right)^n \Phi_s + \left(-\frac{a}{b}\right)^{n-1} + \left(-\frac{a}{b}\right)^{n-2}\frac{1}{\Phi_s} + \cdots + \left(-\frac{a}{b}\right)\frac{1}{\Phi_s^{n-2}}}{\left(-\frac{a}{b}\right)^n + \left(-\frac{a}{b}\right)^{n-1}\frac{1}{\Phi_s} + \left(-\frac{a}{b}\right)^{n-2}\frac{1}{\Phi_s^2} + \cdots + \left(-\frac{a}{b}\right)^2\frac{1}{\Phi_s^{n-2}}} + o(1) \\ &= \Phi_s + o(1)\,.\end{aligned}$$

If $\Phi_s > |b/a|$, then

$$\begin{aligned}\frac{v_{n+1}}{v_n} &= \frac{\left(-\frac{a}{b}\right)^n + \left(-\frac{a}{b}\right)^{n-1} + \cdots + \left(-\frac{a}{b}\right)}{\left(-\frac{a}{b}\right)^n + \left(-\frac{a}{b}\right)^{n-1} + \left(-\frac{a}{b}\right)^2} + o(1) \\ &= -\frac{b}{a} + o(1)\,.\end{aligned}$$

□

3. Proof of Lemma 2.1

First, we shall see the Lemma holds for $s = 2$ and $s = 3$.

Let G_n be the general Fibonacci numbers defined by $G_n = G_{n-1} + G_{n-2}$ $(n \geq 3)$ with arbitrary initial values G_1 and G_2. Let α be the positive real root, and β be its conjugate of the equation $x^2 - x - 1 = 0$. Note that $\alpha + \beta = 1$ and $\alpha\beta = -1$. By $G_n = (\alpha+\beta)G_{n-1} - \alpha\beta G_{n-2}$, we can have

$$\begin{aligned} G_n &= G_2\left(\frac{\alpha^{n-1}}{\alpha-\beta} + \frac{\beta^{n-1}}{\beta-\alpha}\right) - G_1\left(\frac{\beta\cdot\alpha^{n-1}}{\alpha-\beta} + \frac{\alpha\cdot\beta^{n-1}}{\beta-\alpha}\right) \\ &= \frac{G_2(\alpha^{n-1}-\beta^{n-1}) + G_1(\alpha^{n-2}-\beta^{n-2})}{\alpha-\beta} \qquad (n \geq 1)\,. \end{aligned}$$

Hence, by $|\beta/\alpha| < 1$

$$\begin{aligned} \frac{G_{n+1}}{G_n} &= \frac{G_2(\alpha^n-\beta^n) + G_1(\alpha^{n-1}-\beta^{n-1})}{G_1(\alpha^{n-1}-\beta^{n-1}) + G_1(\alpha^{n-1}-\beta^{n-1})} \\ &= \frac{G_2(\alpha-(\beta/\alpha)^{n-1}\beta) + G_1(1-(\beta/\alpha)^{n-1})}{G_2(1-(\beta/\alpha)^{n-1}1) + G_1(1/\alpha-(\beta/\alpha)^{n-1}/\beta)} \\ &\to \frac{G_2\alpha + G_1}{G_2 + G_1/\alpha} = \alpha \quad (n \to \infty) \end{aligned}$$

unless $G_2 + G_1/\alpha = 0$ or $G_2 - G_1\beta = 0$.

Let T_n be the general Tribonacci numbers defined by $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ $(n \geq 4)$ with arbitrary initial values T_1, T_2 and T_3. Let α be the real root, β and γ be the imaginary roots of the equation $x^3 - x^2 - x - 1 = 0$. Since

$$T_n - \gamma T_{n-1} = (\alpha+\beta)(T_{n-1} - \gamma T_{n-2}) - \alpha\beta(T_{n-2} - \gamma T_{n-3})\,,$$

by putting $U_n = T_n - \gamma T_{n-1}$ we have

$$U_n = (\alpha+\beta)U_{n-1} - \alpha\beta U_{n-2}\,.$$

As U_n has the same form as G_n, we obtain

$$\begin{aligned} U_n &= T_n - \gamma T_{n-1} \\ &= T_3\left(\frac{\alpha^{n-2}}{\alpha-\beta} + \frac{\beta^{n-2}}{\beta-\alpha}\right) - T_2\left(\frac{\beta\cdot\alpha^{n-2}}{\alpha-\beta} + \frac{\alpha\cdot\beta^{n-2}}{\beta-\alpha}\right). \end{aligned}$$

By

$$\begin{aligned} T_n - \gamma^{n-1}T_1 = &\ \frac{T_3-\gamma T_2}{\alpha-\beta}\left(\frac{\alpha^{n-1}-\gamma^{n-1}}{\alpha-\gamma} - \frac{\beta^{n-1}-\gamma^{n-1}}{\beta-\gamma}\right) \\ &- \frac{T_2-\gamma T_1}{\alpha-\beta}\left(\frac{\beta(\alpha^{n-1}-\gamma^{n-1})}{\alpha-\gamma} - \frac{\alpha(\beta^{n-1}-\gamma^{n-1})}{\beta-\gamma}\right), \end{aligned}$$

we finally have for $n \geq 1$

$$\begin{aligned}
T_n = T_3 &\left(\frac{\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \\
- T_2 &\left(\frac{(\beta+\gamma)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(\gamma+\alpha)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(\alpha+\beta)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)} \right) \\
+ T_1 &\left(\frac{\beta\gamma \cdot \alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\gamma\alpha \cdot \beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\alpha\beta \cdot \gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)} \right).
\end{aligned}$$

Hence by $|\beta/\alpha| < 1$ and $|\gamma/\alpha| < 1$

$$\begin{aligned}
\frac{T_{n+1}}{T_n} = &\Bigg(T_3 \left((\beta-\gamma)\alpha + (\gamma-\alpha)\beta \left(\frac{\beta}{\alpha}\right)^{n-1} + (\alpha-\beta)\gamma\left(\frac{\gamma}{\alpha}\right)^{n-1} \right) \\
&- T_2 \left((\beta+\gamma)(\beta-\gamma)\alpha + (\gamma+\alpha)(\gamma-\alpha)\beta \left(\frac{\beta}{\alpha}\right)^{n-1} + (\alpha+\beta)(\alpha-\beta)\gamma\left(\frac{\gamma}{\alpha}\right)^{n-1} \right) \\
&+ T_1 \left(\beta\gamma(\beta-\gamma)\alpha + \gamma\alpha(\gamma-\alpha)\beta \left(\frac{\beta}{\alpha}\right)^{n-1} + \alpha\beta(\alpha-\beta)\gamma\left(\frac{\gamma}{\alpha}\right)^{n-1} \right) \Bigg) \\
&\Bigg/ \Bigg(T_3 \left((\beta-\gamma) + (\gamma-\alpha) \left(\frac{\beta}{\alpha}\right)^{n-1} + (\alpha-\beta)\left(\frac{\gamma}{\alpha}\right)^{n-1} \right) \\
&- T_2 \left((\beta+\gamma)(\beta-\gamma) + (\gamma+\alpha)(\gamma-\alpha) \left(\frac{\beta}{\alpha}\right)^{n-1} + (\alpha+\beta)(\alpha-\beta)\left(\frac{\gamma}{\alpha}\right)^{n-1} \right) \\
&+ T_1 \left(\beta\gamma(\beta-\gamma) + \gamma\alpha(\gamma-\alpha) \left(\frac{\beta}{\alpha}\right)^{n-1} + \alpha\beta(\alpha-\beta)\left(\frac{\gamma}{\alpha}\right)^{n-1} \right) \Bigg) \\
&\to \frac{T_3(\beta-\gamma)\alpha - T_2(\beta+\gamma)(\beta-\gamma)\alpha + T_1\beta\gamma(\beta-\gamma)\alpha}{T_3(\beta-\gamma) - T_2(\beta+\gamma)(\beta-\gamma) + T_1\beta\gamma(\beta-\gamma)} \\
&= \frac{T_3\alpha - T_2(\beta+\gamma)\alpha + T_1\beta\gamma\alpha}{T_3 - T_2(\beta+\gamma) + T_1\beta\gamma} = \alpha \quad (n \to \infty)
\end{aligned}$$

unless $T_3 - T_2(\beta+\gamma) + T_1\beta\gamma = 0$.

In general, we can prove the following Proposition.

Proposition 3.1. *Let $\chi_1, \chi_2, \ldots, \chi_s$ be the (real and complex) roots of the equation $x^s - x^{s-1} - x^{s-2} - \cdots - x - 1 = 0$. Let $\{u_n\}$ be the sequence defined by $u_n = u_{n-1} + u_{n-2} + \cdots + u_{n-s}$*

($n \geq s+1$) with initial values $u_1, u_2, \ldots, u_s$. Then for $n \geq 1$

$$
\begin{aligned}
u_n = u_s &\left(\frac{\chi_1^{n-1}}{(\chi_1-\chi_2)(\chi_1-\chi_3)\ldots(\chi_1-\chi_s)} + \frac{\chi_2^{n-1}}{(\chi_2-\chi_1)(\chi_2-\chi_3)\ldots(\chi_2-\chi_s)} \right.\\
&\left. + \cdots + \frac{\chi_s^{n-1}}{(\chi_s-\chi_1)(\chi_s-\chi_2)\cdots(\chi_s-\chi_{s-1})} \right)\\
- u_{s-1} &\left(\frac{(\chi_2+\chi_3+\cdots+\chi_s)\chi_1^{n-1}}{(\chi_1-\chi_2)(\chi_1-\chi_3)\ldots(\chi_1-\chi_s)} + \frac{(\chi_1+\chi_3+\cdots+\chi_s)\chi_2^{n-1}}{(\chi_2-\chi_1)(\chi_2-\chi_3)\cdots(\chi_2-\chi_s)} \right.\\
&\left. + \cdots + \frac{(\chi_1+\chi_2+\cdots+\chi_{s-1})\chi_s^{n-1}}{(\chi_s-\chi_1)(\chi_s-\chi_2)\ldots(\chi_s-\chi_{s-1})} \right)\\
+ u_{s-2} &\left(\frac{(\chi_2\chi_3+\cdots+\chi_2\chi_s+\chi_3\chi_4+\cdots+\chi_{s-1}\chi_s)\chi_1^{n-1}}{(\chi_1-\chi_2)(\chi_1-\chi_3)\ldots(\chi_1-\chi_s)} \right.\\
&\left. + \cdots + \frac{(\chi_1\chi_2+\cdots+\chi_1\chi_{s-1}+\chi_2\chi_3+\cdots+\chi_{s-2}\chi_{s-1})\chi_s^{n-1}}{(\chi_s-\chi_1)(\chi_s-\chi_2)\cdots(\chi_s-\chi_{s-1})} \right)\\
- \cdots &\\
+ (-1)^{s-2}u_2 &\left(\frac{(\chi_2\chi_3\cdots\chi_{s-1}+\chi_2\chi_3\cdots\chi_{s-2}\chi_s+\cdots+\chi_3\chi_4\cdots\chi_s)\chi_1^{n-1}}{(\chi_1-\chi_2)(\chi_1-\chi_3)\cdots(\chi_1-\chi_s)} \right.\\
&\left. + \cdots + \frac{(\chi_1\chi_2\cdots\chi_{s-2}+\chi_1\chi_2\cdots\chi_{s-3}\chi_{s-1}+\cdots+\chi_2\chi_3\ldots\chi_{s-1})\chi_s^{n-1}}{(\chi_s-\chi_1)(\chi_s-\chi_2)\cdots(\chi_s-\chi_{s-1})} \right)\\
+ (-1)^{s-1}u_1 &\left(\frac{\chi_2\chi_3\cdots\chi_s\cdot\chi_1^{n-1}}{(\chi_1-\chi_2)(\chi_1-\chi_3)\cdots(\chi_1-\chi_s)} \right.\\
&\left. + \cdots + \frac{\chi_1\chi_2\cdots\chi_{s-1}\cdot\chi_s^{n-1}}{(\chi_s-\chi_1)(\chi_s-\chi_2)\cdots(\chi_s-\chi_{s-1})} \right).
\end{aligned}
$$

Proof. We have already seen that the statement holds for $s = 2, 3$. We shall show that the statement where s is replaced by $s+1$ holds. Let $\lambda_1, \lambda_2, \ldots, \lambda_{s+1}$ be the (real and complex) roots of the equation $x^{s+1} - x^s - \cdots - x - 1 = 0$. Let $\{w_n\}$ be the sequence defined by $w_n = w_{n-1} + w_{n-2} + \cdots + w_{n-s-1}$ ($n \geq s+2$) with initial values $w_1, w_2, \ldots, w_{s+1}$. Since

$$
\begin{aligned}
w_n - \lambda_{s+1}w_{n-1} &= (\lambda_1+\cdots+\lambda_s)(w_{n-1}-\lambda_{s+1}w_{n-2})\\
&\quad - (\lambda_1\lambda_2+\cdots+\lambda_{s-1}\lambda_s)(w_{n-2}-\lambda_{s+1}w_{n-3})\\
&\quad + \cdots + (-1)^{s+1}\lambda_1\cdots\lambda_s(w_{n-s}-\lambda_{s+1}w_{n-s-1}),
\end{aligned}
$$

$w_n - \lambda_{s+1}w_{n-1}$ has the same form as u_n. Hence,

$$
\begin{aligned}
&w_n - \lambda_{s+1}w_{n-1}\\
&= (w_{s+1}-\lambda_{s+1}w_s)\left(\frac{\lambda_1^{n-2}}{(\lambda_1-\lambda_2)\ldots(\lambda_1-\lambda_s)} + \cdots + \frac{\lambda_s^{n-2}}{(\lambda_s-\lambda_1)\ldots(\lambda_s-\lambda_{s-1})} \right)\\
&\quad - (w_s-\lambda_{s+1}w_{s-1})\left(\frac{(\lambda_2+\cdots+\lambda_s)\lambda_1^{n-2}}{(\lambda_1-\lambda_2)\ldots(\lambda_1-\lambda_s)} + \cdots + \frac{(\lambda_1+\cdots+\lambda_{s-1})\lambda_s^{n-2}}{(\lambda_s-\lambda_1)\ldots(\lambda_s-\lambda_{s-1})} \right)\\
&\quad + \cdots\\
&\quad + (-1)^{s+1}(w_2-\lambda_{s+1}w_1)\left(\frac{\lambda_2\ldots\lambda_s\cdot\lambda_1^{n-2}}{(\lambda_1-\lambda_2)\ldots(\lambda_1-\lambda_s)} + \cdots + \frac{\lambda_1\ldots\lambda_{s-1}\cdot\lambda_s^{n-2}}{(\lambda_s-\lambda_1)\ldots(\lambda_s-\lambda_{s-1})} \right).
\end{aligned}
$$

By multiplying $1, \lambda_{s+1}, \dots, \lambda_{s+1}^{n-2}$ on the both sides, respectively, and adding all equalities side by side, we obtain

$$\begin{aligned}
w_n - \lambda_{s+1}^{n-1} w_1 = (w_{s+1} - \lambda_{s+1} w_s) \Bigg(& \frac{\lambda_1^{n-1} - \lambda_{s+1}^{n-1}}{(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_s)(\lambda_1 - \lambda_{s+1})} \\
& + \cdots + \frac{\lambda_s^{n-1} - \lambda_{s+1}^{n-1}}{(\lambda_s - \lambda_1)\cdots(\lambda_s - \lambda_{s-1})(\lambda_s - \lambda_{s+1})} \Bigg) \\
& - (w_s - \lambda_{s+1} w_{s-1}) \Bigg(\frac{(\lambda_2 + \cdots + \lambda_s)(\lambda_1^{n-1} - \lambda_{s+1}^{n-1})}{(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_s)(\lambda_1 - \lambda_{s+1})} \\
& + \cdots + \frac{(\lambda_1 + \cdots + \lambda_{s-1})(\lambda_s^{n-1} - \lambda_{s+1}^{n-1})}{(\lambda_s - \lambda_1)\cdots(\lambda_s - \lambda_{s-1})(\lambda_s - \lambda_{s+1})} \Bigg) \\
& + \cdots \\
& + (-1)^{s+1}(w_2 - \lambda_{s+1} w_1) \Bigg(\frac{\lambda_2 \cdots \lambda_s (\lambda_1^{n-1} - \lambda_{s+1}^{n-1})}{(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_s)(\lambda_1 - \lambda_{s+1})} \\
& + \cdots + \frac{\lambda_1 \cdots \lambda_{s-1} (\lambda_s^{n-1} - \lambda_{s+1}^{n-1})}{(\lambda_s - \lambda_1)\cdots(\lambda_s - \lambda_{s-1})(\lambda_s - \lambda_{s+1})} \Bigg),
\end{aligned}$$

entailing that

$$\begin{aligned}
w_n = w_{s+1} \Bigg(& \frac{\lambda_1^{n-1}}{(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_s)(\lambda_1 - \lambda_{s+1})} \\
& + \cdots + \frac{\lambda_s^{n-1}}{(\lambda_s - \lambda_1)\cdots(\lambda_s - \lambda_{s-1})(\lambda_s - \lambda_{s+1})} \\
& + \frac{\lambda_{s+1}^{n-1}}{(\lambda_{s+1} - \lambda_1)\cdots(\lambda_{s+1} - \lambda_{s-1})(\lambda_{s+1} - \lambda_s)} \Bigg) \\
& - w_s \Bigg(\frac{(\lambda_2 + \cdots + \lambda_s + \lambda_{s+1})\lambda_1^{n-1}}{(\lambda_1 - \lambda_2)\cdots(\lambda_1 - \lambda_s)(\lambda_1 - \lambda_{s+1})} \\
& + \cdots + \frac{(\lambda_1 + \cdots + \lambda_{s-1} + \lambda_{s+1})\lambda_s^{n-1}}{(\lambda_s - \lambda_1)\cdots(\lambda_s - \lambda_{s-1})(\lambda_s - \lambda_{s+1})} \\
& + \frac{(\lambda_1 + \cdots + \lambda_{s-1} + \lambda_s)\lambda_{s+1}^{n-1}}{(\lambda_{s+1} - \lambda_1)\cdots(\lambda_{s+1} - \lambda_{s-1})(\lambda_{s+1} - \lambda_s)} \Bigg) \\
& + \cdots \\
& + (-1)^s w_1 \Bigg(\frac{\lambda_2 \cdots \lambda_s \lambda_{s+1} \cdot \lambda_1^{n-1}}{(\lambda_1 - \lambda_2)\dots(\lambda_1 - \lambda_s)(\lambda_1 - \lambda_{s+1})} \\
& + \cdots + \frac{\lambda_1 \cdots \lambda_{s-1} \lambda_{s+1} \cdot \lambda_s^{n-1}}{(\lambda_s - \lambda_1)\cdots(\lambda_s - \lambda_{s-1})(\lambda_s - \lambda_{s+1})} \\
& + \frac{\lambda_1 \cdots \lambda_{s-1} \lambda_s \cdot \lambda_{s+1}^{n-1}}{(\lambda_{s+1} - \lambda_1)\cdots(\lambda_{s+1} - \lambda_{s-1})(\lambda_{s+1} - \lambda_s)} \Bigg).
\end{aligned}$$

□

Proof of Lemma 2.1. Assume that χ_1 is the largest positive root. When n tends to infinity, by $|\chi_i/\chi_1| < 1$ $(i \geq 2)$, only the terms including χ_1^{n-1} are considered in u_{n+1}/u_n, where u_n is given in Proposition 3.1. Therefore, by dividing both the numerator and the denominator by χ_1^{n-1}

$$\begin{aligned}\frac{u_{n+1}}{u_n} \to &\Big(u_s\chi_1 - u_{s-1}(\chi_2 + \cdots + \chi_s)\chi_1 + u_{s-2}(\chi_2\chi_3 + \cdots + \chi_{s-1}\chi_s)\chi_1 \\ &- \cdots + (-1)^{s-2}u_2(\chi_2\cdots\chi_{s-1} + \cdots + \chi_3\cdots\chi_s)\chi_1 + (-1)^{s-1}u_1\cdot\chi_2\cdots\chi_s\cdot\chi_1\Big) \\ &\Big/\Big(u_s - u_{s-1}(\chi_2 + \cdots + \chi_s) + u_{s-2}(\chi_2\chi_3 + \cdots + \chi_{s-1}\chi_s) \\ &- \cdots + (-1)^{s-2}u_2(\chi_2\cdots\chi_{s-1} + \cdots + \chi_3\ldots\chi_s) + (-1)^{s-1}u_1\cdot\chi_2\cdots\chi_s\Big) \\ = &\ \chi_1 \quad (n \to \infty)\end{aligned}$$

unless

$$\begin{aligned}u_s - u_{s-1}(\chi_2 + \cdots + \chi_s) + u_{s-2}(\chi_2\chi_3 + \cdots + \chi_{s-1}\chi_s)& \\ - \cdots + (-1)^{s-2}u_2(\chi_2\cdots\chi_{s-1} + \cdots + \chi_3\cdots\chi_s) + (-1)^{s-1}u_1\cdot\chi_2\cdots\chi_s &= 0.\end{aligned}$$

□

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