REMARKS ON THE "GREEDY ODD" EGYPTIAN FRACTION ALGORITHM II

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Abstract. Let a, b be positive, relatively prime integers with $a < b$ and b odd. Let $1/x_1$ be the greatest Egyptian fraction with x_1 odd and $1/x_1 \leq a/b$. We form the difference $a/b - 1/x_1 =: a_1/b_1$ (with $\gcd(a_1, b_1) = 1$) and, if a_1/b_1 is not zero, continue similarly. Given an odd prime p and $1 < a < p$, we prove the existence of infinitely many odd numbers b such that $\gcd(a, b) = 1$, $a < b$, and the sequence of numerators $a_0 := a, a_1, a_2, \ldots$ is $a, a+1, a+2, \ldots, p-1, 1$.

1. Introduction

We denote the set of positive integers by $\mathbb{N}$, the set of non-negative integers by $\mathbb{N}_0$, and the set of prime numbers by $\mathbb{P}$. Consider $a, b \in \mathbb{N}$ with

$$b \text{ odd}, \quad a < b, \quad \gcd(a, b) = 1. \tag{1.1}$$

The *greedy odd algorithm* (or *odd greedy algorithm*) is defined as follows: we take the greatest Egyptian fraction $1/x_1$ with x_1 odd and $1/x_1 \leq a/b$, form the difference $a/b - 1/x_1 =: a_1/b_1$ (with $\gcd(a_1, b_1) = 1$) and, if a_1/b_1 is not zero, continue similarly. As a result, we get the equation

$$\frac{a}{b} = \frac{1}{x_1} + \frac{1}{x_2} + \cdots . \tag{1.2}$$

A well-known open problem is whether the greedy odd algorithm always stops after finitely many steps, i.e., whether the sum in (1.2) is always finite [2, 3, 4].

This is a direct continuation of the paper [5], from which we have taken enough material so that the mathematics of the present paper can be understood, and for which we refer for background and motivation. The translation [6], a reference to the "normal" greedy algorithm of Fibonacci, had not appeared at the time of the writing of [5].

Let p be an odd prime and let $a \in \mathbb{N}, 1 < a < p$. In this paper, using elementary methods, we prove (see Corollary 3.6) the existence of infinitely many numbers b, satisfying (1.1), such that the *sequence of numerators* (for the fraction a/b) $a_0 := a, a_1, a_2, \ldots$ is $a, a+1, a+2, \ldots, p-1, 1$. The inspiration came from the final remark in [5].

2. Notation and Preliminaries

Using the notation of [5], we write $b =: 2k + 1$ and $x_1 =: 2n_1 + 1$ with $k, n_1 \in \mathbb{N}$. It follows from the definition of the greedy odd algorithm that n_1 is the unique positive integer satifying the condition

$$\frac{1}{2n_1 + 1} \leq \frac{a}{2k + 1} < \frac{1}{2n_1 - 1}. \tag{2.1}$$

We further write

$$\frac{a}{2k+1} - \frac{1}{2n_1+1} =: \frac{a_1'}{(2k+1)(2n_1+1)} =: \frac{a_1'}{2k_1'+1} =: \frac{a_1}{2k_1+1}, \tag{2.2}$$

where $\gcd(a_1, 2k_1+1) = 1$.

It follows that

$$a_i \not\equiv a_{i+1} \pmod 2 \text{ for } i = 0, 1, 2, \ldots \tag{2.3}$$

and

$$k_1' = 2kn_1 + k + n_1. \tag{2.4}$$

We need the following result [5, Cor. 3.3] for which we give here a simple direct proof for the convenience of the reader.

Lemma 2.1. *Let $a > 1$ and $k \in \mathbb{N}, k \equiv -1 \pmod a$. Then*

$$a < 2k+1, \quad \gcd(a, 2k+1) = 1, \quad n_1 = \frac{k+1}{a}, \text{ and } a_1' = a+1. \tag{2.5}$$

Proof. We define $n_1 \in \mathbb{N}$ by $n_1 := (k+1)/a$ and then we have to show that n_1 satisfies (2.1). We show that

$$\left\lceil \frac{2k+1}{a} \right\rceil = 2n_1, \tag{2.6}$$

from which (2.1) immediately follows.

We have $2n_1 = (2k+2)/a = (2k+1)/a + 1/a$, where, by assumption, $0 < 1/a < 1$. This proves (2.6).

It follows from (2.6), since $n_1 \in \mathbb{N}$, that $\lceil (2k+1)/a \rceil \geq 2$, which implies that $(2k+1)/a > 1$, i.e., $a < 2k+1$.

A simple calculation shows that $a_1' = a+1$.

Finally, from $2k+1 \equiv -1 \pmod a$, it follows that $\gcd(a, 2k+1) = 1$. □

Lemma 2.2. *Let $a > 1$ and $n \in \mathbb{N}$ be given, and let $k := -1 + t \cdot a \cdot (a+1) \cdots (a+n)$, where $t \in \mathbb{N}$. Then*

$$k_1' \equiv -1 \pmod{(a+1)\cdots(a+n)}. \tag{2.7}$$

Proof. This follows directly from (2.4), since, by (2.5),

$$n_1 = t \cdot (a+1) \cdots (a+n) \equiv 0 \pmod{(a+1)\cdots(a+n)}.$$

□

Theorem 2.3. *Let $a > 1$ and $n \in \mathbb{N}_0$ be given, and let $k := -1 + t \cdot a \cdot (a+1) \cdots (a+n)$. Then*

(a) *the sequence of numerators for the fraction $a/(2k+1)$ starts with $a, a+1, a+2, \ldots, a+n$,*
(b) *$a_{n+1}' = a+n+1$, and*
(c) *$k_i' = k_i$ for $i = 1, \ldots, n$ $(n \in \mathbb{N})$.*

Proof. (a) We use induction on n.

1) $n = 0$. This is a trivial case, since, by Lemma 2.1, $b := 2k+1$ satisfies (1.1).
2) Suppose that $n \in \mathbb{N}$. Lemma 2.1 implies that $a/(2k+1) - 1/(2n_1+1) = (a+1)/(2k_1'+1)$. It follows from (2.7) that $k_1' \equiv -1 \pmod{a+1}$, so that, using Lemma 2.1 again, we have $\gcd(a+1, 2k_1'+1) = 1$, from which it follows, by (2.2), that $k_1 = k_1' \equiv -1 \pmod{(a+1)\cdots(a+n)}$.

Using our induction hypothesis, we see that the sequence of numerators for the fraction $(a+1)/(2k_1+1)$ starts with $a+1, a+2, \dots, a+n$. It follows now immediately from the definition of the greedy odd algorithm, that the sequence of numerators for the fraction $a/(2k+1)$ starts with $a, a+1, a+2, \dots, a+n$.

(b) and (c) can be proved similarly (and more easily). □

Remark 2.4. *The condition* $k \equiv -1 \pmod{a(a+1)\cdots(a+n)}$ *is not necessary for the result of Theorem 2.3 (a),* [5, Theorem 3.8].

Example 2.5. *We take* $a := 2$ *and* $n := 4$*. For* $1 \le t \le 7$ *we give the sequences of numerators for the fractions* $2/(2k+1)$*, where* $k := -1 + t \cdot 6!$*, in Table 1 below.*

Table 1. Examples of Theorem 2.3 (a) with $a := 2, n := 4$.

t	$a_0, a_1, a_2, \dots$
1	$2, 3, 4, 5, 6, 1.$
2	$2, 3, 4, 5, 6, 7, 2, 1.$
3	$2, 3, 4, 5, 6, 1.$
4	$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1.$
5	$2, 3, 4, 5, 6, 7, 8, 9, 10, 1.$
6	$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1.$
7	$2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, 1.$

3. Sequences of Numerators $a, a+1, a+2, \dots, 2m, 1$

For the rest of this paper, we stay with Theorem 2.3. We are interested in sequences like $2, 3, 4, 5, 6, 1$, appearing two times in Table 1, which clearly are minimal in length. In general, it follows from (2.3) that a sequence of numerators can have the form $a, a+1, a+2, \dots, a+n, 1$ only in the case that the number $a+n$ is even. Therefore, from now on, we will always suppose that

$$a + n =: 2m, \text{ with } m \in \mathbb{N}. \tag{3.1}$$

Lemma 3.1. *The sequence of numerators is* $a, a+1, a+2, \dots, 2m, 1$ *if and only if* $2k'_{n+1}+1 \equiv 0 \pmod{2m+1}$.

Proof. This follows immediately from Theorem 2.3 (b). □

Lemma 3.2. *Let* $a > 1$ *and* $n \in \mathbb{N}$ *be given, and suppose that* $k := -1 + t \cdot a(a+1)\cdots(a+n)$ *and* $K := -1 + T \cdot a(a+1)\cdots(a+n)$*, where* $t, T \in \mathbb{N}$ *satisfy* $T \equiv t \pmod{2m+1}$*. Defining* K'_1*,* $K_1, \dots$ *starting from* K *in the same manner as* k'_1*,* $k_1, \dots$ *are defined starting from* k*, we have*

$$k'_{n+1} \equiv K'_{n+1} \pmod{2m+1}. \tag{3.2}$$

Proof. We know from Lemma 2.2 that $k'_1 = -1 + t^* \cdot (a+1)\cdots(a+n)$, for some $t^* \in \mathbb{N}$. In fact, an easy calculation shows that

$$t^* = 2a(a+1)\cdots(a+n)t^2 + (a-1)t. \tag{3.3}$$

It follows from (3.3) that if we write $K'_1 =: -1 + T^* \cdot (a+1)\cdots(a+n)$, then $T^* \equiv t^* \pmod{2m+1}$. By Theorem 2.3 (c), $k_1 = k'_1$ and $K_1 = K'_1$. Using induction, we obtain (3.2). □

Corollary 3.3. *With k and K as in Lemma 3.2, suppose that the sequence of numerators for the fraction $a/(2k+1)$ is $a, a+1, a+2, \ldots, 2m, 1$. Then the sequence of numerators for $a/(2K+1)$ is also $a, a+1, a+2, \ldots, 2m, 1$.*

Proof. This follows immediately from Lemma 3.1 and Lemma 3.2. □

Example 3.4. *To give an example of Corollary 3.3, we give Table 2 below, which is a continuation of Table 1 for $8 \le t \le 14$. Here $2m+1 = 7$, so we should get the sequence $2, 3, 4, 5, 6, 1$ for the values $t = 8$ and $t = 10$.*

TABLE 2. Continuation of Table 1.

t	$a_0, a_1, a_2, \ldots$
8	$2, 3, 4, 5, 6, 1.$
9	$2, 3, 4, 5, 6, 7, 2, 1.$
10	$2, 3, 4, 5, 6, 1.$
11	$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 2, 3, 4, 1.$
12	$2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, 1.$
13	$2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, 1.$
14	$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 1.$

For some values of $a > 1$ and $2m \ge a$, we do not get (from Theorem 2.3) any sequences of numerators of the form $a, a+1, a+2, \ldots, 2m, 1$. Take for example, $a := 2$ and $2m := 8$. In view of Corollary 3.3, it is enough to calculate the sequences of numerators for the fractions $2/(2k+1)$ when $k := -1 + t \cdot 8!$, $t = 1, \ldots, 9$. We give the results in Table 3 below.

TABLE 3. The case $a = 2$, $a + n = 2m = 8$.

t	$a_0, a_1, a_2, \ldots$
1	$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1.$
2	$2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, 1.$
3	$2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, 1.$
4	$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1.$
5	$2, 3, 4, 5, 6, 7, 8, 9, 10, 1.$
6	$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 2, 3, 4, 1.$
7	$2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, 1.$
8	$2, 3, 4, 5, 6, 7, 8, 9, 2, 3, 4, 1.$
9	$2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 1.$

We see that Table 3 does not contain the sequence $2, 3, 4, 5, 6, 7, 8, 1$. The following result shows that this situation cannot happen, when $2m+1$ is a prime.

Theorem 3.5. *Suppose that $a := 2$ and $2m + 1 =: p \in \mathbb{P} \backslash \{2\}$. The value $t = 1$ gives the sequence of numerators $2, 3, \ldots, p-1, 1$.*

Proof. We have $k = -1 + (p-1)! \equiv -2 \pmod{p}$, using Wilson's Theorem [1]. From Lemma 2.1 we get $n_1 = (k+1)/2 = (p-1)!/2$. Using Wilson's Theorem again, we conclude that

$n_1 \equiv (p-1)/2 \pmod p$. Using (2.4),we get

$$k_1' = 2kn_1 + k + n_1 \equiv 2(-2)\cdot\frac{p-1}{2} + (-2) + \frac{p-1}{2}$$
$$\equiv (-2)(-1) - 2 + \frac{p-1}{2} \equiv \frac{p-1}{2} \pmod p.$$

Since the theorem is obviously true if $2m+1=3$, we may suppose that $n \in \mathbb{N}$, so that, by Theorem 2.3 (c), we get

$$k_1 = k_1' \equiv \frac{p-1}{2} \pmod p. \tag{3.4}$$

We prove, using (3.4), that

$$k_1 \equiv k_2 \equiv \cdots \equiv k_{p-3} \equiv k_{p-2}' \equiv \frac{p-1}{2} = m \pmod p, \tag{3.5}$$

from which the theorem follows, using Lemma 3.1.

Suppose that for some i, $1 \le i \le p-3$, we have $k_i \equiv (p-1)/2 \pmod p$. Then, using (2.4), we have

$$k_{i+1}' = 2k_i n_{i+1} + k_i + n_{i+1} \equiv 2\cdot\frac{p-1}{2}\cdot n_{i+1} + n_{i+1} + k_i$$
$$\equiv -n_{i+1} + n_{i+1} + k_i \equiv k_i \equiv \frac{p-1}{2} \pmod p.$$

If $i < p-3$, then $i+1 \le p-3$, and Theorem 2.3 (c) implies that $k_{i+1}' = k_{i+1}$. So we start with $i=1$, and repeat the argument, thereby establishing (3.5). □

Theorem 3.5 obviously implies the following, seemingly more general result.

Corollary 3.6. *Let $p \in \mathbb{P}\backslash\{2\}$ and let $1 < a < p$. There exists a number $t \in \{1,\dots,p\}$ such that if $k := -1 + t\cdot a(a+1)\cdots(p-1)$, then the sequence of numerators for the fraction $a/(2k+1)$ is $a, a+1, a+2, \dots, p-1, 1$. (If $a := 2$, then we can take $t := 1$.) Moreover, by Corollary 3.3, the same sequence results by using the greedy odd algorithm for all fractions $a/(2K+1)$, where $K := -1 + T\cdot a(a+1)\cdots(p-1)$, and $T \equiv t \pmod p$.*

Example 3.7. *We take $a := 3$, $p := 5$ in Corollary 3.6 and we calculate the sequences of numerators corresponding to the values $1 \le t \le 10$. We give the results in Table 4 below.*

Table 4. The case $a=3$, $p=5$ of Corollary 3.6.

t	$a_0, a_1, a_2, \dots$
1	3, 4, 5, 4, 1.
2	3, 4, 5, 2, 1.
3	3, 4, 1.
4	3, 4, 1.
5	3, 4, 5, 2, 1.
6	3, 4, 5, 4, 1.
7	3, 4, 5, 2, 1.
8	3, 4, 1.
9	3, 4, 1.
10	3, 4, 5, 2, 1.

4. Determining All ‘Solutions’ When $n \in \{0, 1, 2\}$

We might look at Corollary 3.6 as saying that a ‘solution’ to our ‘problem’, namely the problem of finding a number $t \in \{1, \dots, p\}$ giving the sequence $a, a+1, a+2, \dots, p-1, 1$, always exists. Inspecting Table 4 and taking the last part of Corollary 3.6 into account, we might want to say that in the case $a := 3$, $p := 5$, we have two solutions, $t \equiv 3, 4 \pmod 5$. When we write $p-1 = 2m = a+n$ (see (3.1)), we can determine all solutions when $n \in \{0, 1, 2\}$ (without having to compute the sequences of numerators).

We start with the easiest case, $n = 0$.

Theorem 4.1. *Let $p \in \mathbb{P}\backslash\{2\}$, $a := p - 1 = 2m$. The unique solution is $t \equiv m \pmod p$.*

Proof. Let $k := -1 + t \cdot (p-1)$. By Lemma 2.1, we have $n_1 = t = (k+1)/(p-1)$. A short calculation gives

$$\frac{a}{2k+1} - \frac{1}{2n_1+1} = \frac{p}{(2tp-2t-1)(2t+1)},$$

from which we see that the sequence of numerators is $p-1, 1$ if and only if $2t+1 \equiv 0 \pmod p$. The theorem follows. □

Next, we handle the case $n = 1$.

Theorem 4.2. *Let $p \in \mathbb{P}$, $p \geq 5$, $a := p-2$. The solutions are $t = [2]^{-1}$, $t = [4]^{-1} \in \mathbb{Z}/p\mathbb{Z}$, i.e., the solutions of the congruences $2t \equiv 1 \pmod p$, $4t \equiv 1 \pmod p$.*

Proof. Let $k := -1 + t \cdot (p-2)(p-1)$. Using Theorem 2.3 (c) and (3.3) we see, after substituting $n = 1$ and $a = p-2$, that $k_1 = k_1' = -1 + t^*(p-1)$, where

$$t^* = 2(p-2)(p-1)t^2 + (p-3)t \equiv 4t^2 - 3t \pmod p.$$

By Theorem 4.1, we have to find t such that $4t^2 - 3t \equiv m \pmod p$. Since $m = (p-1)/2$, this leads to the congruence

$$8t^2 - 6t + 1 \equiv 0 \pmod p$$

and to the two solutions given in the theorem. (Note that we have $8t^2 - 6t + 1 = 8(t - 1/2)(t - 1/4)$ in $\mathbb{Q}[t]$.) □

Finally, we turn to the case $n = 2$. In principle, this is similar to the previous case. Before stating the result, let us recall the standard method (see, for example, [1]) of solving the general quadratic congruence

$$ax^2 + bx + c \equiv 0 \pmod p, \quad p \in \mathbb{P}\backslash\{2\}, \quad p \not| a. \tag{4.1}$$

We first consider the congruence

$$y^2 \equiv d := b^2 - 4ac \pmod p \tag{4.2}$$

and then, if (4.2) is solvable, solve the congruence

$$2ax + b \equiv y \pmod p. \tag{4.3}$$

Theorem 4.3. *Let $p \in \mathbb{P}$, $p \geq 5$, $a := p-3$. If $p \equiv 5, 7 \pmod 8$, then we have only two solutions, the solutions of*

$$12t + 2 \equiv \pm 1 \pmod p. \tag{4.4}$$

If $p \equiv 1, 3 \pmod 8$, then we have two additional solutions, namely those obtained by first solving

$$y^2 \equiv -32 \pmod p \tag{4.5}$$

and then solving

$$48t + 8 \equiv y \pmod{p}. \tag{4.6}$$

Proof. Proceeding as in the proof of Theorem 4.2, we find that

$$t^* = 2(p-3)(p-2)(p-1)t^2 + (p-4)t \equiv -12t^2 - 4t \pmod{p}.$$

Using Theorem 4.2, this leads to the congruences

$$24t^2 + 8t + 1 \equiv 0 \pmod{p} \tag{4.7}$$

and

$$48t^2 + 16t + 1 \equiv 0 \pmod{p}. \tag{4.8}$$

Using the theory of quadratic residues (see, for example, [1]), we obtain the solutions given in the theorem. □

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