THE “MAGICNESS” OF POWERS OF SOME MAGIC SQUARES

CHARLES K. COOK, MICHAEL R. BACON, AND REBECCA A. HILLMAN

Abstract. Powers of matrices whose elements form semimagic or magic squares are investigated and powers of several examples of classical magic squares are computed. Conditions that guarantee their magic properties (“magicness”) are retained or lost are explored.

1. Introduction

Magic squares have interested mathematicians and puzzle solvers for centuries. There is even an 8×8 magic square attributed to Benjamin Franklin [1, 3, 4, 28]. A variety of interesting discoveries have been made and related research studies have ensued. Several such contributions can be found listed in [29]. A delightful puzzle book containing a chapter on magic and Latin squares with many examples can be found in [16]. The approach here will be to investigate the “magicness” of the powers of a variety of magic and semimagic squares. A few comments about terminology should prove interesting and helpful, and a proposition concerning row and column sums in a square matrix will be used throughout the paper.

Terminology: An $n \times n$ array is called magic if the sums of the rows, columns and the main and secondary diagonals are all equal. If so, the sum N is called the magic constant. The definition usually, but not always, implies that the square is regular; namely, that numbers in the square array run from 1 to n^2. Stark [24, p. 118], for example uses 0 to $n^2 - 1$, and also indicates that using the “staircase” procedure for magic square construction, attributed to De la Loubre [16, p. 4 ff.] and generalized by D. N. Lehmer [24, pp. 10, 118 ff.], one may start with any number.

If the diagonals do not add to the magic constant, the square is called semimagic, and if all of the rows, columns, and diagonals of an array with consecutive integers yield different sums, the square is antimagic.

In a magic square, if the numbers symmetrically opposite the center, $a_{i,j} + a_{n-i+1,n-j+1}$, add to a constant the magic square is called associative (or associated) [13].

Comments: The magic constant, N, of an $n \times n$ magic or semimagic square can be computed in various ways. For example, as seen in the array Z in Section 4, $N =$ (largest entry + smallest entry)$(n/2)$ where the entries need not even be sequential [13, 16]; and if the square is regular, $N = n(n^2 + 1)/2$ [19, p. 85]. Various additional types of magic squares and their properties as well as other magic configurations have been explored but will not be considered here. See for example, [14, 15, 17, 23, 24, 25, 29] and the references cited there. For the purpose of brevity the phrase “matrices whose entries represent semimagic or magic squares” will simply be referred to as semimagic or magic.

Proposition 1.1. *If A and B are $n \times n$ semimagic matrices with magic constants M and N, respectively, then AB and BA are semimagic with magic constant MN.*

Partial support for this paper was provided by the University of South Carolina Sumter.

Proof. Let $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. The sum of the jth column of C is

$$\sum_{i=1}^{n} c_{ij} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}b_{kj} = \sum_{k=1}^{n} Mb_{kj} = M\sum_{k=1}^{n} b_{kj} = MN.$$

The argument for the sum of the ith row is similar. □

It is noted that if A is magic and if some power of A is constant, then since a constant matrix times a magic square is constant, it follows that all higher powers are magic.

2. Powers of 3×3 Magic Square Matrices

Many papers have addressed third order magic squares. See for example, Gauthier [12] and Van den Essen [27] and the works cited there. Some of their work is duplicated here but the approach here varies slightly.

For $n = 3$ and with entries from 1 to $3^2 = 9$, the magic constant is $N = 3(10/2) = 15$. First consider the antimagic square

$$A_0 = \begin{bmatrix} 1 & 2 & 3 \\ 8 & 9 & 4 \\ 7 & 6 & 5 \end{bmatrix},\ A_0^2 = \begin{bmatrix} 38 & 38 & 26 \\ 108 & 121 & 80 \\ 90 & 98 & 70 \end{bmatrix},$$

$$A_0^3 = \begin{bmatrix} 524 & 574 & 396 \\ 1636 & 1785 & 1208 \\ 1364 & 1482 & 1012 \end{bmatrix},\ A_0^4 = \begin{bmatrix} 7888 & 8590 & 5848 \\ 24372 & 26585 & 18088 \\ 20304 & 22138 & 15080 \end{bmatrix}.$$

The row sums for A_0 are 6, 21 and 18; the column sums are 16, 17, and 12; and the diagonal sums are 15 and 19. Those for A_0^2 are 102, 309 and 258; 236, 257 and 176; and 229 and 237, respectively. Those for A_0^3 are 1494, 4629 and 3858; 3524, 3841 and 2616; and 3321 and 3545, respectively. Those for A_0^4 are 22326, 69045 and 57522; 52564, 57313 and 39016; and 49553 and 52737, respectively. In general it appears that powers of an antimagic square will remain antimagic.

Next, since the digit 1 cannot dwell on a diagonal, and using the method of De la Loubre [16], the following semimagic square, for example, can be constructed.

$$A_1 = \begin{bmatrix} 9 & 2 & 4 \\ 1 & 6 & 8 \\ 5 & 7 & 3 \end{bmatrix},\ A_1^2 = \begin{bmatrix} 103 & 58 & 64 \\ 55 & 94 & 76 \\ 67 & 73 & 85 \end{bmatrix},\ A_1^3 = \begin{bmatrix} 1305 & 1002 & 1068 \\ 969 & 1206 & 1200 \\ 1101 & 1167 & 1107 \end{bmatrix}.$$

The next and probably the most famous 3×3 as shown by Chernick [7] and others is magic.

$$A_2 = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix},\ A_2^2 = \begin{bmatrix} 91 & 67 & 67 \\ 67 & 91 & 67 \\ 67 & 67 & 91 \end{bmatrix},\ A_2^3 = \begin{bmatrix} 1197 & 1029 & 1149 \\ 1077 & 1125 & 1173 \\ 1101 & 1221 & 1053 \end{bmatrix}$$

$$A_2^4 = \begin{bmatrix} 17259 & 16683 & 16683 \\ 16683 & 17259 & 16683 \\ 16683 & 16683 & 17259 \end{bmatrix},\ A_2^5 = \begin{bmatrix} 254853 & 250821 & 253701 \\ 251973 & 253125 & 254277 \\ 252549 & 255429 & 251397 \end{bmatrix}.$$

A_1 is semimagic; as are its powers, with magic constants 15, 15^2, and 15^3, respectively. A_2, A_2^3, and A_2^5 are all magic with magic constant 15, 15^3, and 15^5, respectively, but A_2^2 and A_2^4 with magic constants 15^2 and 15^4, respectively, are only semimagic.

Note that Trigg [26] uses A_2, which the Chinese called the lo shu magic square, to generate a variety of antimagic squares. Also note that A_2 (except for rotations, etc.) is the only 3×3 regular magic square, which is not surprising since the center digit for a regular odd magic square must be the median, N/n, [7, 16] which here is $15/3 = 5$. Powers of A_2 have been considered in detail in other papers. For example, see [12, 27] for alternate detailed proofs of the following.

Proposition 2.1. *Let M be an associative 3×3 magic square with magic constant m. Then M^{2n+1} is magic with magic constant m^{2n+1}. But M^{2n} is only semimagic.*

Proof. It follows from Chernick [7] that any 3×3 magic square, M, can be written as $[k] + S$ where $[k] = k \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ and $S = \begin{bmatrix} a & -(a+b) & b \\ -(a-b) & 0 & a-b \\ -b & a+b & -a \end{bmatrix}$ for some constants k, a, and b. Note next that S is magic with magic constant 0, $[k]S = S[k] = [0]$ and that $[k]^n = [(3k)^{n-1}k]$. Thus $M^n = ([k]+S)^n = [k]^n + S^n = [(3k)^{n-1}k] + S^n$. By induction it can be seen that

$$S^{2n} = 3^{n-1}(a^2-b^2)^n \cdot P \text{ and } S^{2n+1} = 3^n(a^2-b^2)^n \cdot S \text{ where } P = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Since magic and semimagic squares are closed under addition it also follows that $M^{2n} = (3k)^{2n-1}[k] + S^{2n} = (3k)^{2n-1}[k] + 3^{n-1}(a^2-b^2)^n P$. Since $[k]$ is magic and P is only semimagic, M^{2n} is only semimagic. Also, $M^{2n+1} = (3k)^{2n}[k] + S^{2n+1} = (3k)^{2n}[k] + 3^n(a^2-b^2)^n S$ and since $[k]$ and S are both magic, M^{2n+1} is always magic. Furthermore, if the magic constant for M^1 is m, it follows from Proposition 1.1 in the introduction that the magic constant for M^n will be m^n. □

For example, for $k \geq 1$, A_2^{2k} is semimagic, with magic constant 15^{2k} and A_2^{2k-1} is magic with magic constant 15^{2k-1}.

Finally note that because S is magic with magic constant 0, any numbers could be used for a and b, including Fibonacci, Jacobsthal, Lucas, Pell or other familiar sequential numbers.

The general question of the existence of magic squares with only distinct Fibonacci numbers was asked by Alfred [2], and later Brown [6], by contradicting the sum of Fibonacci numbers formula, proved that no such magic square can exist. Later Madachy [20] asked for a constructive proof that no such magic square can exist with consecutive Fibonacci numbers and provided a proof [21] by showing that the largest Fibonacci number in such an array must be greater than the magic constant.

3. Powers of 4×4 Magic Square Matrices

First, consider the Freitag magic square involving Fibonacci numbers [11]. She derived a formula for constructing any number of such magic squares, $[F_a]$, having magic constant F_{a+8}.

$$[F_a] = \begin{bmatrix} F_{a+2} & F_{a+6} & F_{a+1}+F_{a+6} & F_{a+4} \\ F_{a+3}+F_{a+6} & F_{a+3} & F_{a+1}+F_{a+5} & F_a+F_{a+4} \\ F_{a+2}+F_{a+5} & F_a+F_{a+6} & F_{a+5} & 2F_{a+1} \\ F_{a+1}+F_{a+4} & F_{a+1}+F_{a+3} & F_a+F_{a+2} & F_{a+7} \end{bmatrix},$$

and exhibited in the example

$$[F_5] = \begin{bmatrix} 13 & 89 & 97 & 34 \\ 110 & 21 & 63 & 39 \\ 68 & 94 & 55 & 16 \\ 42 & 29 & 18 & 144 \end{bmatrix}, \ [F_5]^2 = \begin{bmatrix} 17983 & 13130 & 12815 & 10361 \\ 9662 & 17284 & 16166 & 11183 \\ 15636 & 13660 & 15831 & 9162 \\ 11008 & 10215 & 9483 & 25583 \end{bmatrix}.$$

$[F_5]$ is magic but not associative with magic constant $F_{13} = 233$. $[F_5]^2$ is only semimagic with magic constant $(233)^2$. Likewise $[F_5]^3$ and $[F_5]^4$ are semimagic with magic constants $(233)^3$ and $(233)^4$, respectively. In all three cases both diagonals fail to sum to the magic constant. Note that $[F_5]$ is not associative because, for example, $13 + 144 \neq 34 + 42$. Thus it does not satisfy the criterion of Proposition 3.1 below. However, it does follow from Proposition 1.1 that the kth powers, $k > 2$, of $[F_5]$ are also semimagic with magic constant $(233)^k$.

Note that by using relationships between Fibonacci and Lucas numbers a magic square analogous to $[F_a]$ can be constructed.

Recalling the Pell recursion, $P_{n+2} = 2P_{n+1} + P_n$, with $P_0 = 0$ and $P_1 = 1$, it is possible to construct a Pell-type magic square, $[P_a]$, analogous to Freitag's $[F_a]$.

$$[P_a] = \begin{bmatrix} P_{a+2} & 2P_{a+5} + 4P_{a+4} & 2P_{a+1} + 4P_{a+6} & 2P_{a+3} + 4P_{a+2} \\ 2P_{a+3} + 4P_{a+6} & 2P_{a+3} & 2P_{a+5} + 2P_{a+1} & P_a + 4P_{a+4} \\ 2P_{a+5} + 4P_{a+2} & P_a + 4P_{a+6} & 2P_{a+5} & 4P_{a+1} \\ 2P_{a+1} + 4P_{a+4} & 2P_{a+3} + 2P_{a+1} & P_a + 4P_{a+2} & 2P_{a+7} \end{bmatrix}.$$

The magic constant is seen to be P_{a+8}.

Other Pell-type arrays are possible. For example, interested readers might try to construct one with magic constant P_{a+14}. Additional arrays using any number of familiar sequences could also be constructed.

Next consider the antimagic square

$$B_0 = \begin{bmatrix} 15 & 2 & 12 & 4 \\ 1 & 14 & 10 & 5 \\ 8 & 9 & 3 & 16 \\ 11 & 13 & 6 & 7 \end{bmatrix}, \ B_0^2 = \begin{bmatrix} 367 & 218 & 260 & 290 \\ 164 & 353 & 212 & 269 \\ 329 & 377 & 291 & 237 \\ 303 & 349 & 322 & 254 \end{bmatrix}, \text{ and}$$

$$B_0^3 = \begin{bmatrix} 10993 & 9896 & 9104 & 8748 \\ 7468 & 10675 & 7748 & 7696 \\ 10247 & 11636 & 10013 & 9516 \\ 10264 & 11692 & 9616 & 9887 \end{bmatrix}.$$

The row sums of B_0 are 33, 30, 36 and 37; whereas the column sums are 35, 38, 31 and 32; and, the diagonal sums are 39 and 34. The row sums of B_0^2 are 1135, 998, 1234 and 1228; whereas the column sums are 1163, 1297, 1085 and 1050; and, the diagonal sums are 1265 and 1182. The row sums of B_0^3 are 38741, 33587, 41412 and 41459; whereas the column sums are 38972, 43899, 36481 and 35847; and, the diagonal sums are 41568 and 38396. Again it appears that the powers of antimagic arrays are also antimagic.

The following famous example is attributed to Albrecht Dürer, a sixteenth-century German painter. See [5, 16].

$$B_1 = \begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}, \; B_1^2 = \begin{bmatrix} 341 & 285 & 261 & 269 \\ 261 & 301 & 309 & 285 \\ 285 & 309 & 301 & 261 \\ 269 & 261 & 285 & 341 \end{bmatrix},$$

$$B_1^3 = \begin{bmatrix} 10306 & 9474 & 9410 & 10114 \\ 9602 & 9922 & 9986 & 9794 \\ 9858 & 9666 & 9730 & 10050 \\ 9538 & 10242 & 10178 & 9346 \end{bmatrix}, \; B_1^4 = \begin{bmatrix} 337412 & 333828 & 332292 & 332804 \\ 332292 & 334852 & 335364 & 333828 \\ 333828 & 335364 & 334852 & 332292 \\ 332804 & 332292 & 333828 & 337412 \end{bmatrix},$$

$$B_1^5 = \begin{bmatrix} 11389576 & 11336328 & 11332232 & 11377288 \\ 11344520 & 11365000 & 11369096 & 11356808 \\ 11360904 & 11348616 & 11352712 & 11373192 \\ 11340424 & 11385480 & 11381384 & 11328136 \end{bmatrix}.$$

B_1, B_1^3 and B_1^5 are all magic with magic constant 34, 34^3, and 34^5, respectively; but B_1^2 and B_1^4 with magic constants 34^2 and 34^4, respectively, are only semimagic. Observations of powers of B_1 suggest the following.

Proposition 3.1. *Let M be an associative 4×4 magic square with magic constant m. Then M^{2n+1} is magic with magic constant m^{2n+1}. But M^{2n} is only semimagic.*

The proof is somewhat analogous to that of Proposition 2.1 using the method of Chernick [7] to determine the matrices $[k]$ and S. The matrix $[k]$ is determined from the magic constant of the given magic square, M, while S can be shown to be

$$S = \begin{bmatrix} 2a-2c & -2a+c+d & 2b+c+d & -2b-2d \\ -2b+c-d & 2b & -2a & 2a-c+d \\ -2a+c-d & 2a & -2b & 2b-c+d \\ 2b+2d & -2b-c-d & 2a-c-d & -2a+2c \end{bmatrix},$$

and that S^3 is a constant multiple of S. For example, for $k \geq 1$, B_1^{2k} is semimagic, with magic constant 34^{2k} and B_1^{2k-1} is magic with magic constant 34^{2k-1}.

4. Examples of Powers of Magic Square Matrices of Order 5–7

Consider the non-regular 5×5 magic square [16, p. 8]. Note that the magic number satisfies the requirement (the smallest entry + the largest entry) $(n/2) = (4+36)(5/2) = 100$. However the central number, 27, is not the median, $100/5 = 20$.

$$V = \begin{bmatrix} 4 & 13 & 19 & 28 & 36 \\ 26 & 35 & 8 & 11 & 20 \\ 15 & 18 & 27 & 33 & 7 \\ 34 & 5 & 14 & 22 & 25 \\ 21 & 29 & 32 & 6 & 12 \end{bmatrix}, \; V^2 = \begin{bmatrix} 2347 & 2033 & 2237 & 1714 & 1669 \\ 1928 & 2342 & 1784 & 1739 & 2207 \\ 2202 & 1679 & 1844 & 2277 & 1998 \\ 1749 & 1704 & 2172 & 2103 & 2271 \\ 1774 & 2242 & 1963 & 2167 & 1854 \end{bmatrix}.$$

V is magic with magic constant $N = 100$, but V^2 is only semimagic with V^2 magic constant $(100)^2$. It can be shown that V^3 is magic with magic constant $(100)^3$ and that V^4 is only semimagic with magic constant $(100)^4$.

The 6×6 magic square [16, p. 33] shown next has magic constant $N = 111$, but all of its kth powers can be shown to be only semimagic, and with magic constant $(111)^k$.

$$W = \begin{bmatrix} 31 & 9 & 2 & 22 & 27 & 20 \\ 3 & 32 & 7 & 21 & 23 & 25 \\ 8 & 28 & 6 & 26 & 19 & 24 \\ 35 & 1 & 33 & 17 & 10 & 15 \\ 30 & 5 & 34 & 12 & 14 & 16 \\ 4 & 36 & 29 & 13 & 18 & 11 \end{bmatrix},$$

$$W^2 = \begin{bmatrix} 2664 & 1500 & 2361 & 1881 & 2040 & 1875 \\ 1770 & 2283 & 2472 & 1878 & 1932 & 1986 \\ 1956 & 2121 & 2448 & 1902 & 1932 & 1962 \\ 2307 & 1878 & 1611 & 2253 & 2175 & 2097 \\ 2121 & 2040 & 1635 & 2229 & 2175 & 2121 \\ 1503 & 2499 & 1794 & 2178 & 2067 & 2280 \end{bmatrix}.$$

The following 7×7 example [16, p. 12] shows that an odd order magic square satisfying the conditions of Proposition 1.1 does not imply that an even power cannot be magic. The proposition only guarantees that a product of magic (and hence, semimagic) squares will be semimagic and not necessarily magic.

$$Z = \begin{bmatrix} 59 & 99 & 167 & 11 & 79 & 147 & 187 \\ 91 & 131 & 199 & 43 & 111 & 151 & 23 \\ 95 & 163 & 35 & 75 & 143 & 183 & 55 \\ 127 & 195 & 39 & 107 & 175 & 19 & 87 \\ 159 & 31 & 71 & 139 & 179 & 51 & 119 \\ 191 & 63 & 103 & 171 & 15 & 83 & 123 \\ 27 & 67 & 135 & 203 & 47 & 115 & 155 \end{bmatrix},$$

$$Z^2 = \begin{bmatrix} 75439 & 72415 & 81823 & 92687 & 66591 & 92127 & 79919 \\ 88767 & 81487 & 76447 & 72079 & 80927 & 91231 & 70063 \\ 92463 & 70735 & 88879 & 81039 & 74655 & 75215 & 78015 \\ 76335 & 77791 & 91679 & 69391 & 91679 & 77791 & 76335 \\ 78015 & 75215 & 74655 & 81039 & 88879 & 70735 & 92463 \\ 70063 & 91231 & 80927 & 72079 & 76447 & 81487 & 88767 \\ 79919 & 92127 & 66591 & 92687 & 81823 & 72415 & 75439 \end{bmatrix}.$$

Z and Z^2 are both magic with magic constant 749 and $561001 = (749)^2$, respectively. Z^3 and Z^5 are magic, while Z^4 and Z^6 are semimagic failing to be magic on the main diagonal. But the magic constants are the appropriate powers of 749.

5. Ben Franklin's 8×8 Magic Square Matrix, BF

An informative and entertaining presentation of the history and structuring of Franklin magic squares can be found in [3]. Several variations of the square have been considered. For example, one by Abrahams is found in the "Magic Squares" section of the Suzanne Alejandre Math Forum at Drexel web site [1]. Andrews [4, p. 97 ff.] discusses a 16×16 Franklin magic square variation. The Franklin magic Square displayed here is given by Alejandre in [1] as

$$BF = \begin{bmatrix} 14 & 3 & 62 & 51 & 46 & 35 & 30 & 10 \\ 52 & 61 & 4 & 13 & 20 & 29 & 36 & 45 \\ 11 & 6 & 59 & 54 & 43 & 38 & 27 & 22 \\ 53 & 60 & 5 & 12 & 21 & 28 & 37 & 44 \\ 55 & 58 & 7 & 10 & 23 & 26 & 39 & 42 \\ 9 & 8 & 57 & 56 & 41 & 40 & 25 & 24 \\ 50 & 63 & 2 & 15 & 18 & 31 & 34 & 47 \\ 16 & 1 & 64 & 49 & 48 & 33 & 32 & 17 \end{bmatrix}.$$

Proposition 5.1. *The Alejandre Form of the Ben Franklin* 8×8 *Magic Square and all of its* k*th integral powers are magic, with magic constant* $(260)^k$*.*

The proof follows from the comment indicated after Proposition 1.1 in the introduction and is illustrated by the following comments.

$$(BF)^2 = \begin{bmatrix} 8386 & 8514 & 8386 & 8514 & 8386 & 8514 & 8386 & 8514 \\ 8514 & 8386 & 8514 & 8386 & 8514 & 8386 & 8514 & 8386 \\ 8386 & 8514 & 8386 & 8514 & 8386 & 8514 & 8386 & 8514 \\ 8514 & 8386 & 8514 & 8386 & 8514 & 8386 & 8514 & 8386 \\ 8514 & 8386 & 8514 & 8386 & 8514 & 8386 & 8514 & 8386 \\ 8386 & 8514 & 8386 & 8514 & 8386 & 8514 & 8386 & 8514 \\ 8514 & 8386 & 8514 & 8386 & 8514 & 8386 & 8514 & 8386 \\ 8386 & 8514 & 8386 & 8514 & 8386 & 8514 & 8386 & 8514 \end{bmatrix}.$$

Proof. $(BF)^3$ is the constant matrix [2197000]. Both BF and $(BF)^2$ are magic with magic constants 260 and $(260)^2$, respectively. Since $(BF)^3$ is a constant matrix, it follows that it is magic with magic constant $8(2197000) = 17576000 = (260)^3$. Thus $(BF)^k$ is a constant matrix for $k > 2$, with magic constant $(260)^k$. Finally $(BF)^{k+1}$ is magic with magic constant $(260)^{k+1}$. □

For example, $(BF)^4$ is a constant matrix [571220000] with magic constant $8(571220000) = (260)^4$.

6. The General Case

Proposition 6.1. *Let* M *be any associative* $p \times p$ *magic square with magic constant* m*. Then* M^{2n+1} *is magic with magic constant* m^{2n+1}*, and* M^{2n} *is semimagic.*

Similar to the approaches taken in [7, 12, 25, 27] the proof involves matrix algebra and an analysis of the eigenvalues of the various matrices, thus requiring a different approach than is being emphasized here. Various examples of other matrices involving combinations of familiar sequences as indicated in Section 3 will be addressed in a later paper.

7. Some Additional Comments

As indicated in the introduction, various magic configurations have been investigated and the interested reader might want to check under “Magic Squares” in *The Fibonacci Quarterly* Index [8] on the home page of The Fibonacci Association, and on the Wolfram MathWorld Website with its sundry sources given there under “Magic Squares” [28, 29, 30]. The background and ideas at these sites should provide entertaining reading and possibly lead to further research.

An informative and entertaining place for the novice to start would be the chapters in Kelsey [16]. Puzzles and various entertainments involving magic squares can be found in sources such as [9, p. 119 ff.], [10, pp. 29–30, l49], and [22, pp. 47–56]. Approaches more appealing to the professional mathematician can be found in Stark [24] who provides a theoretical approach to magic squares and in Loly, Cameron, Trump and Schindel [18] who used linear algebra and investigated the eigenvalues of magic squares.

Other related topics, to name a few, include a method for the construction of certain even magic squares [23], the property of balanced magic squares [14], and multiplicative type magic squares where the magic constant is determined, not from the row, column, and diagonal sums, but from their products [30]. Note also that an excellent source of information on various magic configurations, including a detailed analysis of Franklin magic squares, can be found in [4] which has recently been reprinted.

Finally, as an entertaining aside, note that Albrecht Dürer, the magic square B_1 and the Franklin magic square all play a role in the Dan Brown novel, *The Lost Symbol* [5, pp. 256, 263 ff., 389].

8. Conclusion

Powers of several magic and semimagic squares have been computed and observations have been made concerning the “magicness” of these powers. That is to say, are powers of magic squares magic, semimagic, or neither? Many additional cases involving other sequences could be investigated. An analysis of the existence or nonexistence of magic squares consisting of only members of said sequences, similar to those made in [2, 6, 20, 21] could be addressed. Also matrices similar to Freitag’s for other sequences might be possible to construct.

Finally, the authors wish to acknowledge the many comments, suggestions and additional references provided by Curtis Cooper and the anonymous referee. These recommendations greatly contributed to the improvement of the presentation of this paper.

References

[1] N. Abrahams, *The Franklin square*, `http://mathforum.org/alejandre/magic.square/franklin/franklin.square.html.`
[2] Br. U. Alfred, *Exploring magic squares*, The Fibonacci Quarterly, **2.3** (1964), 216.
[3] M. A. Amela, *Structured* 8×8 *Franklin squares*, `http://www.region.com.ar/Amela/FranklinSquares.`
[4] W. S. Andrews, *Magic Squares and Cubes*, Cosimo, Inc., New York, 2004.
[5] D. Brown, *The Lost Symbol*, Doubleday, New York, 2009.
[6] J. L. Brown, *Reply to exploring Fibonacci magic squares*, The Fibonacci Quarterly, **3.2** (1965), 146.
[7] J. Chernick, *Solution of the general magic square*, The American Mathematical Monthly, **45.3** (1938), 172–175.
[8] C. K. Cook, *Index of The Fibonacci Quarterly*, `http://www.mscs.dal.ca/Fibonacci/quarterlyindex.html.`
[9] H. E. Dudeney, *Amusements in Mathematics*, Dover Publications, Inc., New York, 1958.
[10] H. E. Dudeney, *The Canterbury Puzzles*, Fourth Edition, Dover Publications, Inc., New York, 1958.
[11] H. T. Freitag, *A magic square involving Fibonacci numbers*, The Fibonacci Quarterly, **6.1** (1968), 77–80.
[12] N. Gauthier, *Singular matrices applied to* 3×3 *magic squares*, The Mathematical Gazette, **81** (1997), 225–230.
[13] P. Gerdes, *On Lunda-designs and the construction of associated magic squares of order 4p*, The College Journal of Mathematics, **31.3** (2000), 182–188.
[14] H. S. Hahn, *Another property of magic squares*, The Fibonacci Quarterly, **13.3** (1975), 205–208.
[15] E. Karst, *Magic squares consisting of primes in A. P.*, The Fibonacci Quarterly, **8.4** (1970), 317–323.
[16] K. Kelsey, *The Ultimate Book of Number Puzzles*, Dorset, Great Britain, 1992.
[17] D. E. Knuth, *Very magic squares*, The American Mathematical Monthly, **75.3** (1968), 260–264.

[18] P. Loly, I. Cameron, W. Trump and D. Schindel, *Magic square spectra*, Linear Algebra and its Applications, **430** (2009), 2659–2680.
[19] J. S. Madachy, *Madachy's Mathematical Recreations*, Chapter 4, Dover Publications, Inc., New York, 1979.
[20] J. S. Madachy, *Recreational mathematics*, The Fibonacci Quarterly, **6.2** (1968), 162–166.
[21] J. Madachy, *Recreational mathematics*, The Fibonacci Quarterly, **6.4** (1968), 301–303.
[22] J. S. Meyer, *FUN with Mathematics*, Fawcett Publications, Inc., New York, 1957.
[23] J. Rothstein, *A method for constructing singly even magic squares*, The Fibonacci Quarterly, **11.6** (1973), 543–544.
[24] H. M. Stark, *An Introduction to Number Theory*, Markham Pub. Co., Chicago, 1970.
[25] A. C. Thompson, *Odd magic powers*, The American Mathematical Monthly, **101.4** (1994), 339–342.
[26] C. W. Trigg, *Antimagic squares derived from the third-order magic square*, The Fibonacci Quarterly, **12.4** (1974), 387–390.
[27] van den Essen, *Magic squares and linear algebra*, The American Mathematical Monthly, **97.1** (1990), 60–62.
[28] E. W. Weisstein, *Franklin Magic Square*, `http://mathworld.wolfram.com/FranklinMagicSquare.html`.
[29] E. W. Weisstein, *Magic Squares*, `http://mathworld.wolfram.com/MagicSquare.html`.
[30] E. W. Weisstein, *Multiplication Magic Squares*, `http://mathworld.wolfram.com/MultiplicationMagicSquare.html`.

MSC2010: 00A08, 11B36, 97A20

Sumter, SC 29150
E-mail address: charliecook29150@aim.com

Sumter, SC 29150
E-mail address: baconmr@gmail.com

Department of Mathematics, Science, and Engineering, University of South Carolina Sumter, 200 Miller Road, Sumter, SC 29150
E-mail address: hillmanr@uscsumter.edu