PARTIAL SUMS OF GENERATING FUNCTIONS AS POLYNOMIAL SEQUENCES

CLARK KIMBERLING

Abstract. Partial sum polynomials are defined from a generating function. The generating function and the partial sum polynomials of even degree can be represented as a certain kind of linear combination of squares. Of particular interest are the coefficients b_k in such sums. Examples of partial sum polynomials include Fibonacci polynomials of the 2nd kind, defined by $P_n(z) = z^2P_{n-2}(z) + zP_{n-1}(z) + 1$, with $P_0(z) = 1$ and $P_1(z) = 1 + z$.

1. Introduction

Consider the generating function $F(z) = (1 - z - z^2)^{-1}$ of the Fibonacci numbers:

$$F(z) = 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + \cdots . \tag{1}$$

The partial sums of $F(z)$ comprise the following sequence of polynomials:

$$P_n(z) = \sum_{k=0}^{n} F_{k+1}z^k \tag{2}$$

which satisfy the recurrence

$$P_n = z^2P_{n-2} + zP_{n-1} + 1. \tag{3}$$

For $n \geq 0$, we shall call P_n the *nth Fibonacci polynomial of the 2nd kind*. Much more generally, an arbitrary generating function

$$f(z) = \sum_{k=0}^{\infty} a_kz^k \tag{4}$$

has partial sums which we shall call *partial sum polynomials of* f, (or *of the sequence* $(a_0, a_1, a_2, \ldots)$):

$$\begin{aligned} p_0(z) &= a_0 \\ p_1(z) &= a_0 + a_1z \\ p_2(z) &= a_0 + a_1z + a_2z^2 \\ p_3(z) &= a_0 + a_1z + a_2z^2 + a_3z^3 \\ &\vdots \end{aligned}$$

and which have generating function

$$\frac{1}{f(zt)(1-t)}.$$

If $a_0 = 2$ and $a_1 = 1$, the polynomial p_n will be called the *nth Lucas polynomial of the 2nd kind*; these satisfy the recurrence $P_n = z^2P_{n-2} + zP_{n-1} + 2$. (Recall that the Fibonacci and Lucas polynomials [of the 1st kind] are defined by the recurrence

$$\rho_n = z\rho_{n-1} + \rho_{n-2}, \tag{5}$$

where $\rho_0 = 1$ and $\rho_1 = z$ in the Fibonacci case, and $\rho_0 = 2$ and $\rho_1 = z$ in the Lucas case.)

The purpose of this article is to present a few properties of generating function polynomials p_n, with special attention to the Fibonacci and Lucas polynomials of the 2nd kind.

2. Linear Combinations of Squares

The term "linear combination" is used here to apply to infinite sums as well as finite. We shall show that a generating function (4), under certain mild conditions, is a linear combination of squares, and that the same is true for polynomials of even degree.

Theorem 1. *Let $a = (a_0, a_1, a_2, \ldots)$ be a sequence of nonzero complex numbers, with generating function*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots . \tag{6}$$

Define $b_0 = a_0$ and $c_0 = \dfrac{a_1}{2b_0}$, and assume that $a_2 \neq b_0 c_0^2$, so that the number

$$b_1 = a_2 - b_0 c_0^2 = a_2 - \frac{a_1^2}{4b_0}$$

is not zero. Inductively, define

$$b_k = a_{2k} - \frac{a_{2k-1}^2}{4b_{k-1}} = a_{2k} - b_{k-1}c_{k-1}^2; \tag{7}$$

$$c_k = \frac{a_{2k+1}}{2b_k} \tag{8}$$

assuming at each stage that $a_{2k} \neq b_{k-1}c_{k-1}^2$. Then

$$f(z) = b_0(1 + c_0 z)^2 + b_1 z^2 (1 + c_1 z)^2 + b_2 z^4 (1 + c_2 z)^2 + \cdots . \tag{9}$$

Proof. Expand (9) and compare coefficients with (6). □

Clearly the series (9) has the same convergence interval as (6); for the special case (1), the convergence interval is $[1 - \tau, \tau - 1)$, where $\tau = (1 + \sqrt{5})/2$, the golden ratio.

We can also start with (9) and easily find that $a_0 = b_0$ and

$$a_{2k+1} = 2b_k c_k \quad \text{and} \quad a_{2k+2} = b_{k+1} + b_k c_k^2 \tag{10}$$

for $k \geq 0$.

Example 1. *If $a = (1, 2, 3, 4, \ldots)$, then $b(a) = a$ and $c = (1, 1, 1, 1, \ldots)$.*

As a second example in which the three sequences a, b, c are quite simple, we have the following.

Example 2. *If $a = (1, 1, 1, 1, \ldots)$, then*

$$b_k = \frac{k+2}{2k+2} \quad \text{and} \quad c_k = \frac{k+1}{k+2}$$

for $k \geq 0$.

Theorem 2. *In addition to the hypothesis of Theorem 1, suppose that the following limits exist:*

$$\alpha = \lim_{m\to\infty} \frac{a_{m+1}}{a_m}, \qquad \beta = \lim_{m\to\infty} \frac{b_{m+1}}{b_m}, \qquad \gamma = \lim_{m\to\infty} c_m$$

and that $\gamma \neq 0$. Then $\beta = \alpha^2$ and $\gamma = \alpha$.

Proof. The equations (10), adapted as

$$a_{2k} = b_k + b_{k-1}c_{k-1}^2, \qquad a_{2k+1} = 2b_kc_k, \qquad a_{2k+2} = b_{k+1} + b_kc_k^2,$$

imply

$$\frac{a_{2k+1}}{a_{2k}} = \frac{2b_kc_k}{b_k + b_{k-1}c_{k-1}^2} \qquad \text{and} \qquad \frac{a_{2k+2}}{a_{2k+1}} = \frac{b_{k+1} + b_kc_k^2}{2b_kc_k},$$

so that

$$\alpha = \frac{2\beta\gamma}{\beta+\gamma^2} = \frac{\beta+\gamma^2}{2\gamma},$$

so that $\beta = \alpha^2$ and $\gamma = \alpha$. □

We turn now to an arbitrary even-degree polynomial

$$p_{2n}(z) = a_0 + a_1z + a_2z^2 + \cdots + a_{2n}z^{2n}.$$

The method of Theorem 1 leads to the following linear combination of squares:

$$p_{2n}(z) = b_0(1+c_0z)^2 + b_1z^2(1+c_1z)^2 + \cdots + b_{n-1}z^{2n-2}(1+c_{n-1}z)^2 + b_nz^{2n}, \tag{11}$$

where the finite sequences b and c are given by (7) and (8).

Example 3. *Linear combinations of squares for three Fibonacci polynomials of the 2nd kind are shown here:*

$$\begin{aligned}
F_2(z) &= 1 + z + 2z^2 \\
&= (1+\tfrac{1}{2}z)^2 + \tfrac{7}{4}z^2 \\
F_4(z) &= 1 + z + 2z^2 + 3z^3 + 5z^4 \\
&= (1+\tfrac{1}{2}z)^2 + \tfrac{7}{4}z^2(1+\tfrac{6}{7}z)^2 + \tfrac{26}{7}z^4 \\
F_6(z) &= 1 + z + 2z^2 + 3z^3 + 5z^4 + 8z^5 + 13z^6 \\
&= (1+\tfrac{1}{2}z)^2 + \tfrac{7}{4}z^2(1+\tfrac{6}{7}z)^2 + \tfrac{26}{7}z^4(1+\tfrac{14}{13}z)^2 + \tfrac{113}{13}z^6.
\end{aligned}$$

3. The Case $a = (x, y, x+y, x+2y, \ldots)$

In this section we study the sequences b and c when the given sequence is a generalized Fibonacci sequence—that is, x and y are arbitrary positive numbers, and

$$a_0 = x, \quad a_1 = y, \quad a_2 = x+y, \quad \ldots, \quad a_k = xF_{k-1} + yF_k.$$

This sequence is the classical Fibonacci or Lucas sequence according as $(x,y) = (1,1)$ or $(x,y) = (2,1)$. Of particular interest is the sequence b_k defined in Theorem 1, and I am indebted to Paul Bruckman for an insightful proof of the convergence of b_{k+1}/b_k in the case $(x,y) = (1,1)$. Bruckman's method has served as a guide throughout this section. To begin, define

$$d_k = 1 - \frac{a_{2k+1}}{4b_k} \tag{12a}$$

for $k \geq 0$, and note that this definition yields the following recurrence for the sequence (d_k):

$$1 - d_{k+1} = \frac{a_{2k+3}}{4a_{2k} + 4d_k a_{2k+1}}. \tag{13}$$

We shall need a few technical lemmas about Fibonacci numbers and their relation to the golden ratio, given by

$$\tau = \frac{1+\sqrt{5}}{2} = \lim_{m\to\infty} \frac{F_{m+1}}{F_m}.$$

It will be helpful (e.g., in Lemmas 3 and 5) to define $a_{-1} = y - x$ and $a_{-2} = 2x - y$.

Lemma 1. *If $k \geq 0$, then*

$$F_{2k+3} + F_{2k+1} - \tau F_{2k+1} - 2\tau F_{2k} > 0. \tag{14}$$

Proof. Let $\alpha = \tau$ and $\beta = 1 - \tau$. Let $L_n = \alpha^n + \beta^n$, the nth Lucas number. Since $\beta < \alpha$, we have

$$\frac{L_{2k+2}}{L_{2k+1}} = \frac{\alpha^{2k+2} + \beta^{2k+2}}{\alpha^{2k+1} + \beta^{2k+1}} > \alpha,$$

which implies (14) because $L_m = F_{m-1} + F_{m+1}$ for $m \geq 1$. □

Lemma 2. *For $k \geq 0$, let*

$$s_k = 2\tau F_{2k+1} - \tau F_{2k} - F_{2k} - F_{2k+2} \tag{15}$$

for $k \geq 0$. The sequence (s_k) is strictly decreasing.

Proof. By Lemma 1,

$$\begin{aligned} 0 &> \tau F_{2k+1} + 2\tau F_{2k} - F_{2k+1} - F_{2k+3} \\ &= 2\tau F_{2k+1} + 2\tau F_{2k} - \tau F_{2k+1} - F_{2k+1} - F_{2k+3} \\ &= 2\tau F_{2k+2} - \tau F_{2k+1} - F_{2k+1} - F_{2k+3} \\ &= 2\tau(F_{2k+3} - F_{2k+1}) - \tau(F_{2k+2} - F_{2k}) \\ &\quad -(F_{2k+2} - F_{2k}) - (F_{2k+4} - F_{2k+2}), \end{aligned}$$

so that $s_{k+1} < s_k$. □

Lemma 3. *Suppose that $0 < y \leq \tau x$ and $k \geq 0$. Then*

$$\frac{xF_{2k+2} + yF_{2k+3}}{4(xF_{2k-1} + yF_{2k}) + (4 - 2\tau)(xF_{2k} + yF_{2k+1})} < \tau/2. \tag{16}$$

Proof. It is easy to check that (16) holds for $k = 0$. Assume that $k \geq 1$. By Lemma 2, the sequence $(2\tau F_{2k+1} - \tau F_{2k} - F_{2k} - F_{2k+2})$ is a strictly decreasing sequence of positive numbers. Consequently, for positive x and y,

$$0 < x(2\tau F_{2k+1} - \tau F_{2k} - F_{2k} - F_{2k+2}) + y(2\tau F_{2k+2} - \tau F_{2k+1} - F_{2k+1} - F_{2k+3}),$$

from which easily follows

$$\begin{aligned} 0 < \; & x(4\tau F_{2k+1} + 4\tau F_{2k} - 2\tau^2 F_{2k} - 2F_{2k+2}) \\ & + y(4\tau F_{2k+2} + 4\tau F_{2k+1} - 2\tau^2 F_{2k+1} - 2F_{2k+3}), \end{aligned}$$

whence

$$2xF_{2k+2} + 2yF_{2k+3} < 4\tau xF_{2k-1} + 4\tau yF_{2k} + (4\tau - 2\tau^2)(xF_{2k} + yF_{2k+1}),$$

so that (16) holds for $k \geq 1$. □

Lemma 4. *Suppose that* $0 < y \leq \tau x$ *and* $k \geq 0$. *Then*

$$1 - \tau/2 < d_k. \tag{17}$$

Proof. Clearly (17) holds for $k = 0$. Suppose for arbitrary $k \geq 0$ that $1 - \tau/2 < d_k$. Then

$$\begin{aligned} 1 - d_{k+1} &= \frac{a_{2k+3}}{4a_{2k} + 4d_k a_{2k+1}} \\ &\leq \frac{a_{2k+3}}{4a_{2k} + 4(1-\tau/2)a_{2k+1}} \quad \text{by the induction hypothesis} \\ &\leq \frac{xF_{2k+2} + yF_{2k+3}}{4(xF_{2k-1} + yF_{2k}) + (4-2\tau)(xF_{2k} + yF_{2k+1})}, \end{aligned}$$

so that $1 - d_{k+1} < \tau/2$ by Lemma 3. □

Lemma 5. *If* $k \geq 0$, *then*

$$\begin{aligned} F_{2k+1}F_{2k+4} - F_{2k+2}F_{2k+3} &= 1 \\ 4(F_{2k+1}F_{2k+5} - F_{2k+3}^2 + F_{2k+4}F_{2k+1} - F_{2k+2}F_{2k+3}) + 3(F_{2k+4}F_{2k} - F_{2k+2}^2) &= 5 \\ F_{2k+2}F_{2k+3} - F_{2k+1}F_{2k+4} + 4(F_{2k+5}F_{2k+1} - F_{2k+3}^2) + 3(F_{2k+5}F_{2k} - F_{2k+3}F_{2k+2}) &= -3 \\ F_{2k+3}^2 - F_{2k+1}F_{2k+5} &= -1. \end{aligned}$$

Proof. These identities are all easily proved by induction. □

Lemma 6. *Suppose that* $0 < y \leq \tau x$, *and for* $k \geq 0$, *let*

$$G_k = \frac{x^2F_{2k+2} + xyF_{2k+3}}{4x^2F_{2k+1} + xy(4F_{2k+1} + 3F_{2k}) - y^2F_{2k+1}}.$$

The sequence (G_k) *is strictly increasing.*

Proof. Suppose that $k \geq 0$. The inequality $G_k < G_{k+1}$ to be proved is easily recast as $V - U > 0$, where

$$\begin{aligned} U &= (x^2F_{2k+2} + xyF_{2k+3})(4x^2F_{2k+3} + xy(4F_{2k+3} + 3F_{2k+2}) - y^2F_{2k+3}) \\ V &= (x^2F_{2k+4} + xyF_{2k+5})(4x^2F_{2k+1} + xy(4F_{2k+1} + 3F_{2k}) - y^2F_{2k+1}). \end{aligned}$$

Expanding $V - U$ and using identities in Lemma 5 gives

$$\begin{aligned} V - U &= 4x^4 + 5x^3y - 3x^2y^2 - xy^3 \\ &= x(4x + y)(xy + x^2 - y^2), \end{aligned}$$

which is positive for $0 < y \leq \tau x$. □

Lemma 7. *Suppose that* $0 < y \leq \tau x$ *and* $k \geq 0$. *Then*

$$d_k \leq d_0. \tag{18}$$

Proof. Clearly (18) holds for $k = 0$. Assume that (18) for arbitrary $k \geq 0$. Then

$$4xd_k \leq 4x - y \qquad \text{and} \qquad 4yd_k \leq 4y - y^2/x,$$

so that

$$\begin{aligned} 4d_k(xF_{2k} + yF_{2k+1}) + (y^2/x)F_{2k+1} &\leq (4x - y)F_{2k} + 4yF_{2k+1} \\ &\leq 4xF_{2k} + 4y(F_{2k+1} + 2F_{2k}). \end{aligned}$$

Consequently,

$$\begin{aligned}&4xF_{2k-1}+4yF_{2k}+4d_k(xF_{2k}+yF_{2k+1})\\ &<4xF_{2k+1}+y(4F_{2k+1}+3F_{2k})-(y^2/x)F_{2k+1},\end{aligned}$$

so that

$$\begin{aligned}&\frac{xF_{2k+2}+yF_{2k+3}}{4xF_{2k+1}+y(4F_{2k+1}+3F_{2k})-(y^2/x)F_{2k+1}}\\ &<\frac{xF_{2k+2}+yF_{2k+3}}{4(xF_{2k+1}+yF_{2k})+4d_k(xF_{2k}+yF_{2k+1})},\end{aligned}$$

which is to say that $G_k \le 1-d_{k+1}$. Therefore, by Lemma 6,

$$d_{k+1} \le 1-G_0. \tag{19}$$

Next, the obvious inequality

$$y(y-2x)^2+4x^3>0$$

is equivalent to

$$4y(16x^2-8xy+y^2)+16x(x-y)(4x-y)+16x^2(2y-3x)>0,$$

so that

$$4(\tfrac{4x-y}{4x})^2y+(4x-4y)(\tfrac{4x-y}{4x})+x+2y-4x>0,$$

which is restated as

$$4d_0^2a_1+(4x-4y)d_0+a_3-4x>0,$$

so that

$$1-\frac{a_3}{4x+4d_0a_1}<d_0,$$

which is restated as

$$1-G_0<d_0. \tag{20}$$

Inequalities (19) and (20) yield $d_{k+1} \le d_0$, so that by induction, (18) holds for all $k \ge 0$. □

Lemma 8. *If $k \ge 0$, then*

$$F_{2k-3}F_{2k+2}-F_{2k-1}F_{2k}=2 \tag{21}$$

$$F_{2k-3}F_{2k+3}+F_{2k-2}F_{2k+2}-F_{2k-1}F_{2k+1}-F_{2k}^2=2 \tag{22}$$

$$F_{2k-2}F_{2k+3}-F_{2k}F_{2k+1}=-2 \tag{23}$$

$$F_{2k}^2-F_{2k-2}F_{2k+2}=1 \tag{24}$$

$$2F_{2k}F_{2k+1}-F_{2k-2}F_{2k+3}-F_{2k-1}F_{2+2}=1 \tag{25}$$

$$F_{2k+1}^2-F_{2k-1}F_{2k+3}=-1. \tag{26}$$

Proof. These identities are all easily proved by induction. □

Lemma 9. *Suppose that $x>0$ and $y>0$, and $k \ge 0$. Let*

$$\begin{aligned}E&=x^2+xy-y^2\\ E_1&=a_{2k-2}a_{2k+3}-a_{2k}a_{2k+1}\\ E_2&=a_{2k+1}^2-a_{2k-1}a_{2k+3}.\end{aligned}$$

Then $E_1=2E$ and $E_2=E$.

Proof. When E_1 is expanded using $a_m = xF_{m-1} + yF_m$ for the indicated subscripts m, the result is the sum $q_1x^2 + q_2xy + q_3y^2$ where q_1, q_2, and q_3 are the numbers $2, 2, -2$ given in (21)–(23). Likewise, $E_2 = q_4x^2 + q_5xy + q_6y^2$ where q_4, q_5, and q_6 are the numbers $1, 1, -1$ given in (24)–(26). □

Lemma 10. *If $0 < y \le \tau x$, then the sequence (d_k) is strictly decreasing.*

Proof. Suppose $k \ge 1$. Using E_1 and E_2 as in Lemma 9, we find

$$\begin{aligned}\frac{d_k - d_{k+1}}{4(1-d_k)(1-d_{k+1})} &= \frac{E_1 - d_kE_2 + (d_{k-1} - d_k)a_{2k-1}a_{2k+3}}{a_{2k+1}a_{2k+3}} \\ &= \frac{(2-d_k)D + (d_{k-1} - d_k)a_{2k-1}a_{2k+3}}{a_{2k+1}a_{2k+3}}, \end{aligned} \tag{27}$$

where $D = x^2 + xy + y^2$. Since $d_0 = 1 - y/(4x) < 1$, we have $d_k < 1$ and $d_{k+1} < 1$, by Lemma 7. Accordingly, $(1-d_k)(1-d_{k+1}) > 0$ and $2 - d_k > 0$. As a first induction step, clearly $d_0 > d_1$, and if $d_{k-1} > d_k$ for arbitrary k, then (27) establishes that $d_k > d_{k+1}$. □

Theorem 3. *Suppose that $0 < y \le \tau x$. Then*

$$\lim_{k\to\infty} d_k = 1 - \tau/2 \quad \textit{and} \quad \lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \tau^2.$$

Moreover,

$$a_{2k} < b_{k+1} < a_{2k+2} \tag{28}$$

for $k \ge 0$.

Proof. By Lemmas 4, 7, and 10, the sequence (d_k) is bounded and strictly decreasing. Therefore it converges. Let $d = \lim_{k\to\infty} d_k$. By (13),

$$1 - d_{k+1} = \frac{x\frac{F_{2k+2}}{F_{2k-1}} + y\frac{F_{2k+3}}{F_{2k-1}}}{4(x + y\frac{F_{2k}}{F_{2k-1}}) + 4d_k(x\frac{F_{2k}}{F_{2k-1}} + y\frac{F_{2k+1}}{F_{2k-1}})},$$

so that

$$\begin{aligned}1 - d &= \frac{x\tau^3 + y\tau^4}{4(x + y\tau) + 4d(x\tau + y\tau^2)} \\ &= \frac{\tau^3}{4 + 4d\tau},\end{aligned}$$

which yields $d = 1 - \tau/2$. From (12a) we have

$$\frac{b_{k+1}}{b_k} = \frac{a_{2k+3}}{a_{2k+1}},$$

so that

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = \lim_{k\to\infty} \frac{xF_{2k+2} + yF_{2k}}{xF_{2k} + yF_{2k-2}} = \frac{x\tau^4 + y\tau^2}{x\tau^2 + y} = \tau^2.$$

Next, (7) and (12a) give

$$\begin{aligned}b_{k+1} &= a_{2k+2} - a_{2k+1}(1 - d_k) \\ &= a_{2k} + d_ka_{2k+1},\end{aligned}$$

so that (28) holds, since $0 < d_k < 1$. □

If $a = (F_{k+1})$, then $F_{2k+1} < b_{k+1} < F_{2k+3}$ for $k \geq 0$, by Theorem 3. Experimentation suggests a tighter upper bound, $b_{k+1} < L_{2k+1}$, as well as the inequalities

$$\tau^2 - \frac{1}{k+1} < b_{k+1}/b_k < \tau^2$$

for $k \geq 0$.

If $a = (L_{2k})$ then $L_{2k} < b_{k+1} < L_{2k+2}$ for $k \geq 0$, by Theorem 3, and experimentation suggests that $F_{2k+2} < b_{k+1} < F_{2k+3}$ for $k \geq 2$, and that

$$\tau^2 - \frac{1}{k} < b_{k+1}/b_k < \tau^2$$

for $k \geq 1$.

MSC2000: 11B39, 40A05

Department of Mathematics, University of Evansville, 1800 Lincoln Avenue, Evansville, Indiana 47722

E-mail address: ck6@evansville.edu