SUMS OF SECOND ORDER LINEAR RECURRENCES

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Abstract. This paper examines second order linear homogeneous recurrence relations with coefficients in finite rings. The first section determines conditions under which such sequences are purely periodic. The second section focuses on sums of second order linear recurrences over $\mathbf{Z}_p$, where p is an odd prime. In particular, the question of when the sum of two uniformly distributed second order sequences over $\mathbf{Z}_p$ is uniformly distributed is answered.

1. Purely Periodic Second Order Linear Recurrences

Throughout this section R is a finite local commutative ring with maximal ideal M and multiplicative identity 1_R. Let $\mathbf{Z}$ denote the integers and $\mathbf{N}$ denote the nonnegative integers.

A sequence $\mathbf{s} = \{s_0, s_1, \ldots\}$ of elements in R is purely periodic with period n, if n is the smallest positive integer for which $s_{n+i} = s_i$ for all $i \in \mathbf{N}$. Assume $\mathbf{s}$ is generated by the second order linear recurrence $s_{n+2} = as_{n+1} + bs_n$ for all $n \in \mathbf{N}$, where a, b, s_0, s_1 are fixed elements of R. The characteristic polynomial corresponding to this sequence is given by $f(x) = x^2 - ax - b$. In this section, we establish conditions in terms of a, b, s_0, and s_1 which determine when the corresponding sequence is purely periodic.

The first case we will examine is when $b \in R - M$, that is, b is a unit in R. We appeal to two theorems from McKenzie and Overbay [5]. The first theorem gives a factorization of $f(x)$ in an extension ring and the second uses this factorization to establish a bound on the period of the resulting purely periodic sequence.

Theorem 1.1. *Let $f(x) = x^2 - ax - b \in R[x]$ where $b \in R - M$. Then there exists a ring S which contains R as a subring, an element $r_1 \in S$, and an element $r_2 \in R[r_1]$ such that $f(x) = x^2 - ax - b = (x - r_1)(x - r_2)$.*

Proof. See Theorem 2.1 [5]. □

We note that the product of r_1 and r_2 is b, which is a unit. Thus, the roots of f are units in this case. We let $|r_1|$ denote the order of the element r_1. This is the smallest positive integer for which $r_1^{|r_1|} = 1_R$. We write n_R to denote the characteristic of the R. The next theorem establishes that the sequence $\mathbf{s}$ with characteristic polynomial $f(x)$ is purely periodic and gives bounds on the period of $\mathbf{s}$.

Theorem 1.2. *Let S be a ring which contains R as a subring and let r_1 and r_2 be units in S such that $f(x) = (x - r_1)(x - r_2)$. Let λ be the least common multiple of $|r_1|$ and $|r_2|$. Then $\mathbf{s}$ is purely periodic with period dividing $\lambda \cdot n_R$.*

Proof. See Theorem 2.3 [5]. □

Putting these together, we obtain the following theorem.

Theorem 1.3. *Let $f(x) = x^2 - ax - b \in R[x]$ be the characteristic polynomial of the sequence $\mathbf{s}$ in R, where b is a unit in R. Then $\mathbf{s}$ is purely periodic.*

It is interesting to note that when b is a unit, the sequence $\mathbf{s}$ corresponding to f will always be purely periodic regardless of whether s_0, s_1, and a are units. Furthermore, this includes the cases where f has a repeated root or is irreducible in $R[x]$.

The case where b is not a unit, that is $b \in M$, is more difficult to analyze. We first consider the subcase where $a \in M$, so a and b are both non-units. We obtain the following result.

Theorem 1.4. *Let $f(x) = x^2 - ax - b \in R[x]$ be the characteristic polynomial of the sequence $\mathbf{s}$ in R, where $a, b \in M$. Then $\mathbf{s}$ is purely periodic if and only if $s_i = 0$ for all $i \in \mathbf{N}$.*

Proof. Regardless of the choices for s_0 and s_1 it is clear that $s_2 \in M$ since $s_2 = as_1 + bs_0$, where $a, b \in M$. Similarly, $s_3 = as_2 + bs_1 \in M$. Since $s_2, s_3 \in M$, then $s_4 = as_3 + bs_2 \in M^2$ and $s_5 = as_4 + bs_3 \in M^2$. More generally, $s_i \in M^{\lfloor i/2 \rfloor}$. Since R is a finite local ring with maximal ideal M, then there exists a positive integer k for which $M^k = \{0\}$. Hence, for all $i \in \mathbf{N}$ with $i \geq 2$, if $\lfloor i/2 \rfloor \geq k$, then $s_i = 0$. Thus, $\mathbf{s}$ will only be purely periodic if $s_0 = s_1 = 0$, forcing $s_i = 0$ for all $i \in \mathbf{N}$. □

This theorem applies when f can be factored in $R[x]$ as $f(x) = (x - r_1)(x - r_2)$, where $r_1, r_2 \in M$. Here we have $b = -r_1 r_2 \in M$ and $a = r_1 + r_2 \in M$. Hence, $\mathbf{s}$ will be purely periodic if and only if $s_0 = s_1 = 0$. The theorem also applies in the case where $a, b \in M$, but f does not have any zeros in M. Consider $f(x) = x^2 - 3x - 3$ in $\mathbf{Z_9}[\mathbf{x}]$. In this example we have $a = b = 3 \in M$, but f does not have any zeros in M.

Next we consider the subcase where $b \in M$ and $a \in R - M$. To determine if $\mathbf{s}$ is purely periodic, we also need to examine the initial conditions. When the initial conditions consist of one unit and one non-unit, the resulting sequence is not purely periodic as we see in the following theorem.

Theorem 1.5. *Let $f(x) = x^2 - ax - b \in R[x]$ be the characteristic polynomial of the sequence $\mathbf{s}$ in R, where $b \in M$ and $a \in R - M$. If exactly one of s_0 and s_1 is in M, then $\mathbf{s}$ is not purely periodic.*

Proof. Suppose $s_0 \in M$ and $s_1 \in R - M$. Then $bs_0 \in M$ and $as_1 \in R - M$. The sum of these two, s_2, is also in $R - M$. Now for $i \geq 3$, $s_i \in R - M$ since it is formed by adding an element of M with an element of $R - M$. Since s_0 was not a unit, it follows that $\mathbf{s}$ is not purely periodic.

Similarly, if $s_0 \in R - M$ and $s_1 \in M$, we have both $bs_0 \in M$ and $as_1 \in M$. Now the remaining terms of the sequence are linear combinations of elements of M. Hence, $s_i \in M$ for $i \geq 2$. Since s_0 was a unit, $\mathbf{s}$ is not purely periodic. □

When $b \in M$, $a \in R - M$ and both s_0 and s_1 are in M, the resulting sequence consists entirely of elements of M. However, this sequence may or may not be purely periodic as the following example illustrates.

Example 1.6. *Let $f(x) = x^2 - x - 6 \in \mathbf{Z_9}[\mathbf{x}]$. If $s_0 = 6$ and $s_1 = 3$, then the resulting sequence is given by $6, 3, 3, 3, \ldots$. However, if we change the initial conditions to $s_0 = 3$ and $s_1 = 3$, the resulting sequence is $3, 3, 3, 3, \ldots$.*

Similarly, when $b \in M$, $a \in R - M$ and both s_0 and s_1 are in $R - M$, the resulting sequence consists entirely of units. Again, this sequence may or may not be purely periodic, depending on the choice of initial conditions.

Example 1.7. *Let $f(x) = x^2 - x - 6 \in \mathbf{Z_9}[\mathbf{x}]$. Let $s_0 = 1$ and $s_1 = 1$, then the resulting sequence is given by $1, 1, 7, 4, 1, 7, 4, \ldots$. If the initial conditions are $s_0 = 1$ and $s_1 = 7$, the resulting sequence is $1, 7, 4, 1, 7, 4, \ldots$.*

In these examples, $f(x) = x^2 - x - 6 = (x-3)(x-7)$ has one zero in M and one in $R - M$. This helps us characterize the sequences by the types of zeros of f. In particular, when the zeros of f are both units, then $\mathbf{s}$ is purely periodic. When the zeros of f are both non-units, then $\mathbf{s}$ is not purely periodic unless it is the zero sequence. Finally, if the zeros of f consist of one unit and one non-unit, then the resulting sequence is possibly purely periodic.

2. Sums of Second Order Linear Recurrences

In this section, $R \simeq \mathbf{Z}_p$ where p is an odd prime. We let $s_{n+2} = as_{n+1} + bs_n$ for all $n \in \mathbf{N}$, where $a, b \neq 0, s_0, s_1 \in R$ be the second order linear recurrence over R with characteristic polynomial $f(x) = x^2 - ax - b$. Since $b \neq 0$ and R is a field, it follows that b is a unit. Hence, the sequence $\mathbf{s}$ is purely periodic. Much is known about the distribution properties of second order linear recurrences over finite fields (see for example [1, 2, 4, 6, 7, 8, 9, 10]). In particular, we have the following result of Niederreiter and Shiue.

Theorem 2.1. *Let* $\mathbf{s}$ *be a uniformly distributed second order linear recurrence over a finite field of odd order. Let* $f(x)$ *be the characteristic polynomial associated with* $\mathbf{s}$*, then* $f(x)$ *must have a multiple root.*

Proof. This immediately follows from Corollary 3 [8]. □

Now consider the polynomial $g(x) = bx^2 + ax - 1$ corresponding to the second order linear recurrence as defined above. Note that i is a zero of $g(x)$ if and only if i^{-1} is a zero of $f(x) = x^2 - ax - b$. Hence, if the sequence corresponding to $g(x)$ is uniformly distributed, then $g(x) = b(x-i)^2$, where i is a unit in R.

When $g(x) = b(x-i)^2$, it is not necessarily true that $\mathbf{s}$ is uniformly distributed. When we have such a factorization, it is true that the period of $\mathbf{s}$ divides $p(p-1)$ [8, Lemma 3]. Since $\mathbf{s}$ is purely periodic, it is sufficient to consider the first $p(p-1)$ terms to determine if the sequence is uniformly distributed.

Consider the polynomial $\sum_{j=0}^{m-1} s_j x^j \in R[x]$, where the coefficient s_j is the jth term of the sequence $\mathbf{s}$. We now appeal to the following theorem to express the first m terms of $\mathbf{s}$ as the coefficients of a finite generating function.

Theorem 2.2. *Let* $\mathbf{s}$ *be the sequence generated by the second order recurrence* $s_{n+2} = as_{n+1} + bs_n$ *for all* $n \in \mathbf{N}$*, where* $a, b \neq 0, s_0, s_1 \in R$*. Let* $g(x) = bx^2 + ax - 1 = b(x-i)^2$ *and* $h(x) = (s_1 - as_0)x + s_0$*. Let* $m = p(p-1)$*. Then* $\sum_{j=0}^{m-1} s_j x^j = \frac{h(x)\cdot(x^m - 1)}{g(x)}$.

Proof. See Theorem 1.2 [5]. □

The remainder of this section is devoted to examining the sum of two sequences in R of the form $\frac{(\alpha x+\beta)\cdot(x^m-1)}{(x-i)^2}$. Any uniformly distributed sequence generated by a second order linear recurrence corresponds to such a rational function where $\alpha x + \beta = b^{-1}h(x)$.

Theorem 2.3. *Let* $m = p(p-1)$ *and let* $i \neq 0$*,* $j \neq 0$*,* $\alpha, \beta, \delta,$ *and* γ *be elements of* R*. Then the coefficients of* $\frac{(\alpha x+\beta)\cdot(x^m-1)}{(x-i)^2} + \frac{(\delta x+\gamma)\cdot(x^m-1)}{(x-j)^2}$ *are uniformly distributed in* R *if and only if* $i^k(\alpha i + \beta) \neq -j^k(\delta j + \gamma)$ *for all* $k \in \mathbf{N}$.

Proof. We note that $x^m - 1 = [(x-1)(x-2)\dots(x-(p-1))]^p$, where $1, 2, \dots, (p-1)$ are the distinct nonzero elements of R. Now we have:

$$\begin{aligned} x^m - 1 &= [(x-1)(x-2)\dots(x-(p-1))]^p \\ &= (x-1)^p(x-2)^p\dots(x-(p-1))^p \\ &= (x^p-1)(x^p-2)\dots(x^p-(p-1)). \end{aligned}$$

Let $r \in R$, then

$$\frac{x^p - r}{(x-r)^2} = x^{p-2} + 2rx^{p-3} + 3r^2x^{p-4} + \dots + (p-2)r^{p-3}x + (p-1)r^{p-2}$$

and

$$\frac{x^m - 1}{x^p - r} = x^{p(p-2)} + rx^{p(p-3)} + r^2x^{p(p-4)} + \dots + r^{p-3}x^p + r^{p-2}.$$

Hence,

$$\begin{aligned} \frac{x^m - 1}{(x-r)^2} &= (x^{p-2} + 2rx^{p-3} + 3r^2x^{p-4} + \dots + (p-2)r^{p-3}x + (p-1)r^{p-2}) \\ &\quad \times (x^{p(p-2)} + rx^{p(p-3)} + r^2x^{p(p-4)} + \dots + r^{p-3}x^p + r^{p-2}) \\ &= \Big(\sum_{t=0}^{p-2}(t+1)r^t x^{p-2-t}\Big) \cdot \Big(\sum_{s=0}^{p-2} r^s x^{p(p-2-s)}\Big). \end{aligned}$$

Now

$$\begin{aligned} &\frac{(\alpha x + \beta)\cdot(x^m-1)}{(x-i)^2} + \frac{(\delta x + \gamma)\cdot(x^m-1)}{(x-j)^2} \\ &= (\alpha x + \beta)\Big(\sum_{t=0}^{p-2}(t+1)i^t x^{p-2-t}\Big) \cdot \Big(\sum_{s=0}^{p-2} i^s x^{p(p-2-s)}\Big) \\ &\quad + (\delta x + \gamma)\Big(\sum_{t=0}^{p-2}(t+1)j^t x^{p-2-t}\Big) \cdot \Big(\sum_{s=0}^{p-2} j^s x^{p(p-2-s)}\Big). \end{aligned}$$

The resulting polynomial has degree $m-1$ and consists of m terms. The m coefficients of this polynomial are represented by $(w+1)(\alpha i^{k+1} + \delta j^{k+1}) + w(\beta i^k + \gamma j^k)$ for all $w \in \{0, 1, \dots, p-1\}$ and all $k \in \{0, 1, \dots, p-2\}$. Now fix $k \in \{0, 1, \dots, p-2\}$ and let $u, w \in \{0, 1, \dots, p-1\}$ with $u \neq w$. We note that for any fixed k, if for all $u, w \in \{0, 1, \dots, p-1\}$ with $u \neq w$,

$$(u+1)(\alpha i^{k+1} + \delta j^{k+1}) + u(\beta i^k + \gamma j^k) \neq (w+1)(\alpha i^{k+1} + \delta j^{k+1}) + w(\beta i^k + \gamma j^k),$$

then each element of R occurs exactly once as a coefficient for this particular k value. Further, should this hold for all $k \in \{0, 1, \dots, p-2\}$, then each element of R will appear exactly $p-1$ times within the first m terms. Hence, the sequence is uniformly distributed. Thus, we consider the case where

$$(u+1)(\alpha i^{k+1} + \delta j^{k+1}) + u(\beta i^k + \gamma j^k) = (w+1)(\alpha i^{k+1} + \delta j^{k+1}) + w(\beta i^k + \gamma j^k),$$

where k is fixed and $u \neq w$.

Now we have:

$$(u+1)(\alpha i^{k+1} + \delta j^{k+1}) + u(\beta i^k + \gamma j^k) = (w+1)(\alpha i^{k+1} + \delta j^{k+1}) + w(\beta i^k + \gamma j^k),$$

which implies

$$(u-w)(\alpha i^{k+1} + \delta j^{k+1}) + (u-w)(\beta i^k + \gamma j^k) = 0.$$

Since $u \neq w$, this can be simplified as

$$\alpha i^{k+1} + \delta j^{k+1} + \beta i^k + \gamma j^k = 0,$$

which reduces to

$$i^k(\alpha i + \beta) = -j^k(\delta j + \gamma).$$

Since the steps are reversible, it follows that if $i^k(\alpha i + \beta) = -j^k(\delta j + \gamma)$ for some fixed value of k, then $(u+1)(\alpha i^{k+1} + \delta j^{k+1}) + u(\beta i^k + \gamma j^k) = (w+1)(\alpha i^{k+1} + \delta j^{k+1}) + w(\beta i^k + \gamma j^k)$ for arbitrary $u, w \in \{0, 1, \ldots, p-1\}$ with $u \neq w$. Thus, all p coefficients of the form $(w+1)(\alpha i^{k+1} + \delta j^{k+1}) + w(\beta i^k + \gamma j^k)$ are equal for some k. Hence, at least one element of R appears more than $p-1$ times, so the coefficients are not uniformly distributed. □

The last theorem tells us when the sum of two rational functions of the form $\frac{(\alpha x+\beta)\cdot(x^m-1)}{(x-i)^2}$ has uniformly distributed coefficients. Each of the two separate rational functions which form this sum may or may not have uniformly distributed coefficients. This gives us three possibilities to consider. Before we do this, we want to consider the special case were one of the two rational functions is zero.

We let $\delta = \gamma = 0$. In this case, the coefficients of $\frac{(\alpha x+\beta)\cdot(x^m-1)}{(x-i)^2}$ will be uniformly distributed if and only if $\alpha i + \beta \neq 0$. In other words, $g(x)$ has a zero i of multiplicity two, but it is not a zero of $\alpha x + \beta$. We will show that this is equivalent to known conditions for uniform distribution given in the following theorem.

Theorem 2.4. *Let* $\mathbf{s}$ *be the sequence given by* $s_{n+2} = as_{n+1} + bs_n$ *for all* $n \in \mathbf{N}$*, where* $a, b, c = s_0, d = s_1 \in R$*. Then* $\mathbf{s}$ *is uniformly distributed over* R *if and only if* $a^2 + 4b = 0$ *and* $ad + 2bc \neq 0$*.*

Proof. See Theorem 2 [10]. □

Note that the conditions $a^2 + 4b = 0$ and $ad + 2bc \neq 0$ force both a and b not to be zero.

Theorem 2.5. *Let* $\mathbf{s}$ *be the sequence given by* $s_{n+2} = as_{n+1} + bs_n$ *for all* $n \in \mathbf{N}$*, where* $a, b \neq 0, c = s_0, d = s_1 \in R_p$*. Let* $g(x) = bx^2 + ax - 1, \alpha = b^{-1}(d - ac)$*, and* $\beta = b^{-1}c$*. The following are equivalent:*

(i) $a^2 + 4b = 0$ *and* $ad + 2bc \neq 0$*;*
(ii) $g(x) = bx^2 + ax - 1 = b(x-i)^2$ *for some* $i \in R - \{0\}$ *and* $\alpha i + \beta \neq 0$*.*

Proof. Applying the quadratic formula to $g(x) = bx^2 + ax - 1$, we see that $i = -a(2b)^{-1} = 2a^{-1}$ is a double zero of $g(x)$ if and only if $a^2 + 4b = 0$.

Also

$$\begin{aligned}\alpha i + \beta = 0 &\Leftrightarrow b^{-1}(d - ac)(2a^{-1}) + b^{-1}c = 0 \\ &\Leftrightarrow 2da^{-1} - 2c + c = 0 \\ &\Leftrightarrow 2ad - a^2c = 0 \\ &\Leftrightarrow 2ad + 4bc = 0 \\ &\Leftrightarrow 2bc + ad = 0.\end{aligned}$$

□

These distribution conditions for second order linear recurrences lead to certain subgroups of $\mathbf{Z}_p \times \mathbf{Z}_p$. Consider $s_{n+2} = as_{n+1} + bs_n$ with $a, b, c = s_0, d = s_1 \in \mathbf{Z}_p$. We have the following theorem of Burke.

Theorem 2.6. *Let p be an odd prime such that p divides $a^2 + 4b$, then the set*

$$H_{(a,b)} = \{(c, d) \mid p \text{ divides } ad + 2bc\}$$

is a subgroup of $\mathbf{Z}_p \times \mathbf{Z}_p$.

Proof. See Proposition 1 [2]. □

When $b \neq 0$, the elements of $H_{(a,b)}$ are the pairs of initial conditions (s_0, s_1) for which $s_{n+2} = as_{n+1} + bs_n$ yields a non-uniformly distributed sequence. Of course, the condition p divides $a^2 + 4b$ implies $g(x) = bx^2 + ax - 1$ has a multiple root and can be factored as $g(x) = b(x - i)^2$.

For a fixed non-zero value of $i \in \mathbf{Z}_p$, it is easy to verify that the set $H_i = \{(\alpha, \beta) \mid \alpha i + \beta = 0\}$ also forms an additive subgroup of $\mathbf{Z}_p \times \mathbf{Z}_p$. It is the cyclic group of order p generated by $(1, -i)$. If $(\alpha, \beta) \in H_i$ the coefficients of $\frac{(\alpha x+\beta)\cdot(x^m-1)}{(x-i)^2}$ will not be uniformly distributed and when $(\alpha, \beta) \notin H_i$ the coefficients will be uniformly distributed.

We observe that sets $H_i - (0, 0)$ for $i = 1, 2, \ldots, p - 1$ form a partition of

$$(\mathbf{Z}_p - \{0\}) \times (\mathbf{Z}_p - \{0\})$$

where (α, β) is related to (δ, γ) if and only if $\alpha\gamma = \beta\delta$. This follows since $\alpha i + \beta = 0 = \delta i + \gamma$ implies that $i = -\beta\alpha^{-1} = -\gamma\delta^{-1}$, which gives us $\alpha\gamma = \beta\delta$.

Example 2.7. *Let $p = 5$, then*

$$H_1 = \{(0, 0), (1, 4), (2, 3), (3, 2), (4, 1)\},$$

$$H_2 = \{(0, 0), (1, 3), (2, 1), (3, 4), (4, 2)\},$$

$$H_3 = \{(0, 0), (1, 2), (2, 4), (3, 1), (4, 3)\},$$

and

$$H_4 = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}.$$

This concludes our discussion of the special case where one of the rational functions in Theorem 2.3 is zero. Each of the two rational function in Theorem 2.3 may or may not have coefficients which are uniformly distributed. We end this paper by considering the various possibilities.

In the case where $\frac{(\alpha x+\beta)\cdot(x^m-1)}{(x-i)^2}$ and $\frac{(\delta x+\gamma)\cdot(x^m-1)}{(x-j)^2}$ both have coefficients that are not uniformly distributed, then $\alpha i + \beta = 0$ and $\delta j + \gamma = 0$, which forces $i^k(\alpha i + \beta) = -j^k(\delta j + \gamma)$ for any k. Thus, the sum will not be uniformly distributed.

When the coefficients of both rational functions are uniformly distributed, their sum may or may not have uniformly distributed coefficients. This will depend on whether $i^k(\alpha i + \beta) = -j^k(\delta j + \gamma)$ for some k.

Example 2.8. *Let $p = 5, \alpha = 1, \beta = 0$, and $i = 2$. This gives us the following uniformly distributed sequence:*

$$\frac{(\alpha x+\beta)\cdot(x^m-1)}{(x-i)^2} = \frac{(x)\cdot(x^{20}-1)}{(x-2)^2} = x^{19}+4x^{18}+2x^{17}+2x^{16}+2x^{14} + 3x^{13}+4x^{12}+4x^{11}+4x^9+x^8 + 3x^7+3x^6+3x^4+2x^3+x^2+x.$$

Now let $\delta = 4, \gamma = 0$, and $j = 3$. This gives us another uniformly distributed sequence.

$$\frac{(\delta x+\gamma)\cdot(x^m-1)}{(x-j)^2} = \frac{(4x)\cdot(x^{20}-1)}{(x-3)^2} = 4x^{19}+4x^{18}+3x^{17}+2x^{16}+2x^{14} + 2x^{13}+4x^{12}+x^{11}+x^9+x^8+2x^7 + 3x^6+3x^4+3x^3+x^2+4x.$$

In this case, $i^k(\alpha i+\beta) = -j^k(\delta j+\gamma)$ has a solution when $k = 1$, so the sum of the sequences is not uniformly distributed. In fact,

$$\frac{(x)\cdot(x^{20}-1)}{(x-2)^2} + \frac{(4x)\cdot(x^{20}-1)}{(x-3)^2} = 3x^{18}+4x^{16}+4x^{14}+3x^{12}+2x^8+x^6 + x^4+2x^2.$$

Example 2.9. *Let p, α, β, and i be the same as in the last example. Now let $\delta = 2, \gamma = 0$, and $j = 3$. This gives us the uniformly distributed sequence:*

$$\frac{(\delta x+\gamma)\cdot(x^m-1)}{(x-j)^2} = \frac{(2x)\cdot(x^{20}-1)}{(x-3)^2} = 2x^{19}+2x^{18}+4x^{17}+x^{16}+x^{14}+x^{13} + 2x^{12}+3x^{11}+3x^9+3x^8+x^7+4x^6 + 4x^4+4x^3+3x^2+2x.$$

In this case, $i^k(\alpha i+\beta) = -j^k(\delta j+\gamma)$ has no solution, so the sum of the sequences is uniformly distributed. In fact,

$$\frac{(x)\cdot(x^{20}-1)}{(x-2)^2} + \frac{(2x)\cdot(x^{20}-1)}{(x-3)^2} = 3x^{19}+x^{18}+x^{17}+3x^{16}+3x^{14}+4x^{13} + x^{12}+2x^{11}+2x^9+4x^8+4x^7+2x^6 + 2x^4+x^3+4x^2+3x.$$

Finally, in the case where the coefficients of one rational function are uniformly distributed and the others are not, we see that one side of the equation

$$i^k(\alpha i+\beta) = -j^k(\delta j+\gamma)$$

is equal to zero and the other is not. Thus, the resulting sum has uniformly distributed coefficients.

We note that when we add two second order linear recurrences of the form $\frac{(\alpha x+\beta)\cdot(x^m-1)}{(x-i)^2}$, the coefficients correspond to a linear recurrence, but the order may be as high as four. In the case where this sum is a fourth order linear recurrence for which the characteristic polynomial has distinct double roots, conditions for uniform distribution are also given in Theorem 4B [8].

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