MORE ON COMBINATIONS OF HIGHER POWERS OF FIBONACCI NUMBERS

R. S. MELHAM

Abstract. The Fibonacci identity

$$F_n^4 - F_{n+1}^4 - 10F_{n+2}^4 - F_{n+3}^4 + F_{n+4}^4 = 6F_{2n+4}^2$$

belongs to a family of identities where each identity contains only one product on the right side. In this paper we give this family together with two other such families. We also state two conjectures that give the form of similar identities. Finally, we give the expansions of L_n^{2m} and F_n^{2m} in terms of Lucas numbers with even subscripts.

1. Introduction

In [4] we presented the following identities:

$$F_n^4 + 4F_{n+1}^4 + 4F_{n+2}^4 + F_{n+3}^4 = 6F_{2n+3}^2, \tag{1.1}$$

$$F_n^4 - 6F_{n+2}^4 - 6F_{n+4}^4 + F_{n+6}^4 = 56F_{2n+6}^2 + 20, \tag{1.2}$$

$$F_n^4 + 19F_{n+3}^4 + 19F_{n+6}^4 + F_{n+9}^4 = 1224F_{2n+9}^2 - 480, \tag{1.3}$$

and

$$F_n^4 - 46F_{n+4}^4 - 46F_{n+8}^4 + F_{n+12}^4 = 20304F_{2n+12}^2 + 8100. \tag{1.4}$$

We then proved the following theorem which has (1.1)-(1.4) as special cases.

Theorem 1.1. *Let n and k be integers. Then*

$$\begin{aligned} F_n^4 &+ \left((-1)^{k+1}L_{2k} + 1\right)F_{n+k}^4 + \left((-1)^{k+1}L_{2k} + 1\right)F_{n+2k}^4 + F_{n+3k}^4 \\ &= F_kL_{2k}F_{3k}F_{2n+3k}^2 + 10(-1)^kF_{k-1}F_k^4F_{k+1}. \end{aligned} \tag{1.5}$$

In [4] we presented two further theorems that are analogous to Theorem 1.1 and gave two conjectures that describe the general form of similar results.

Upon further investigation, we found that there is an abundance of similar families of identities with only *one* product on the right side. In this paper we present a selection of such identities. For larger powers such identities become unwieldy. Consequently, to conserve space, we present only identities where the coefficients on the left display symmetry. Our aim here is to present some relatively simple cases in order to highlight the form of such identities. Furthermore, we give two conjectures that describe the general form of similar identities.

2. Three Families of Identities

Consider the identities

$$F_n^4 - F_{n+1}^4 - 10F_{n+2}^4 - F_{n+3}^4 + F_{n+4}^4 = 6F_{2n+4}^2, \tag{2.1}$$

$$-F_n^4 + 81F_{n+2}^4 - 520F_{n+4}^4 + 81F_{n+6}^4 - F_{n+8}^4 = 216F_{2n+8}^2, \tag{2.2}$$

and

$$F_n^4 - 256F_{n+3}^4 - 4930F_{n+6}^4 - 256F_{n+9}^4 + F_{n+12}^4 = 3264F_{2n+12}^2. \tag{2.3}$$

To obtain (2.1)-(2.3) we assumed the existence of identities of the required form and, upon substituting several values of n, solved the resulting equations to obtain the coefficients. After considering several more such identities we obtained a general result that includes (2.1)-(2.3) as special cases. This result is contained in our first theorem.

Theorem 2.1. *Let n and k be integers. Then*

$$\begin{aligned}(-1)^{k+1}F_kF_n^4 + (-1)^kL_k^3F_{2k}F_{n+k}^4 - F_{3k}\left(L_{4k} + 2(-1)^kL_{2k} + 4\right)F_{n+2k}^4 \\ + (-1)^kL_k^3F_{2k}F_{n+3k}^4 + (-1)^{k+1}F_kF_{n+4k}^4 = 3F_{2k}^2F_{3k}F_{2n+4k}^2.\end{aligned} \tag{2.4}$$

In (2.4) the discovery of the coefficient of F_{n+2k}^4 was made easy when we sought an expansion in terms of Lucas numbers that have even subscripts. This idea was pivotal in our discovery of the lengthier identities that we present here.

In order to conveniently state our next theorem, we define coefficients $a_i = a_i(k)$ as follows:

$$\begin{aligned}
a_0 &= (-1)^{k+1}a_5 = (-1)^k\left(L_{2k} + 3(-1)^k\right);\\
a_1 &= (-1)^{k+1}a_4 = (-1)^{k+1}\left(L_{8k} + 4(-1)^kL_{6k} + 9L_{4k} + 12(-1)^kL_{2k} + 13\right);\\
a_2 &= (-1)^{k+1}a_3 = L_{2k}\left(L_{10k} + 3(-1)^kL_{8k} + 6L_{6k} + 11(-1)^kL_{4k} + 14L_{2k} + 15(-1)^k\right).
\end{aligned}$$

We are now able to state our second theorem.

Theorem 2.2. *Let n and k be integers, and let a_i, $0 \le i \le 5$, be as defined above. Then*

$$\sum_{i=0}^{5} a_iF_{n+ik}^6 = 5L_kF_{3k}F_{4k}F_{5k}F_{2n+5k}^3. \tag{2.5}$$

For our third theorem we define coefficients $a_i = a_i(k)$ as

$$\begin{aligned}
a_0 = a_8 &= F_k^2\left(3L_{8k} + 10(-1)^kL_{6k} + 23L_{4k} + 38(-1)^kL_{2k} + 48\right);\\
a_1 = a_7 &= -L_kF_{2k}L_{3k}F_{4k}\left(3L_{8k} + 4(-1)^kL_{6k} + 12L_{4k} + 12(-1)^kL_{2k} + 22\right);\\
a_2 = a_6 &= (-1)^kF_kL_{2k}^2F_{7k}\Big(3L_{12k} + 10(-1)^kL_{10k} + 25L_{8k} + 50(-1)^kL_{6k}\\
&\quad + 81L_{4k} + 108(-1)^kL_{2k} + 122\Big);\\
a_3 = a_5 &= (-1)^{k+1}L_k^2L_{3k}F_{4k}F_{7k}\Big(3L_{12k} + 6(-1)^kL_{10k} + 13L_{8k} + 18(-1)^kL_{6k}\\
&\quad + 29L_{4k} + 32(-1)^kL_{2k} + 42\Big);\\
a_4 &= F_{5k}F_{7k}\Big(3L_{18k} + 9(-1)^kL_{16k} + 24L_{14k} + 47(-1)^kL_{12k}\\
&\quad + 83L_{10k} + 126(-1)^kL_{8k} + 179L_{6k} + 227(-1)^kL_{4k}\\
&\quad + 263L_{2k} + 262(-1)^k\Big).
\end{aligned}$$

Theorem 2.3. *Let n and k be integers, and let a_i, $0 \le i \le 8$, be as defined above. Then*

$$\sum_{i=0}^{8} a_i F_{n+ik}^8 = 35 L_k F_{3k} F_{4k}^2 F_{5k} F_{6k} F_{7k} F_{2n+8k}^4. \tag{2.6}$$

3. Conjectures Concerning Further Identities

As we stated in the introduction, we have confined our investigation to identities in which the coefficients on the left display symmetry. Our investigations have led to two conjectures concerning the existence of such identities. In the statement of these conjectures the "symmetry" to which we have alluded will be made precise.

Conjecture 3.1. *Let $p > 0$ be an even integer, and let $m > 1$ and k be integers. Then there exist integers $a_i = a_i(p, m, k)$, $0 \le i \le mp$, and an integer $b = b(p, m, k)$ such that*

$$\sum_{i=0}^{mp} a_i(k) F_{n+ik}^{mp} = b F_{mn+pm^2k/2}^p. \tag{3.1}$$

Furthermore, for $0 \le i \le mp/2 - 1$, we have $a_i = a_{mp-i}$.

Conjecture 3.2. *Let $p > 1$ be an odd integer, and let $m > 1$ and k be integers. Then there exist integers $a_i = a_i(p, m, k)$, $0 \le i \le mp - 1$, and an integer $b = b(p, m, k)$ such that*

$$\sum_{i=0}^{mp-1} a_i(k) F_{n+ik}^{mp} = b F_{mn+m(pm-1)k/2}^p. \tag{3.2}$$

Furthermore, for $0 \le i \le \lfloor mp/2 \rfloor - 1$, we have

$$a_i = \begin{cases} -a_{mp-1-i}, & \text{if } m \equiv 0 \pmod 4; \\ (-1)^{\lfloor mp/2 \rfloor (k+m-1)+ikm} a_{mp-1-i}, & \text{if } m \not\equiv 0 \pmod 4. \end{cases}$$

In Conjectures 3.1 and 3.2 the case where $k = 0$ is the trivial case in which all coefficients are zero. In Conjecture 3.2 the symmetry condition on the a_i was not easily forthcoming.

In order to illustrate Conjectures 3.1 and 3.2, we present two instances of each.

For $(p, m, k) = (2, 3, 1)$ an instance of Conjecture 3.1 is

$$F_n^6 - 6F_{n+1}^6 - 58F_{n+2}^6 + 198F_{n+3}^6 - 58F_{n+4}^6 - 6F_{n+5}^6 + F_{n+6}^6 = 120F_{3n+9}^2. \tag{3.3}$$

For $(p, m, k) = (4, 2, 1)$ another instance of Conjecture 3.1 is

$$\begin{aligned} 14F_n^8 &- 417F_{n+1}^8 - 10998F_{n+2}^8 + 25896F_{n+3}^8 + 146510F_{n+4}^8 + 25896F_{n+5}^8 \\ &- 10998F_{n+6}^8 - 417F_{n+7}^8 + 14F_{n+8}^8 = 81900F_{2n+8}^4. \end{aligned} \tag{3.4}$$

For $(p, m, k) = (3, 3, 1)$ an instance of Conjecture 3.2 is

$$\begin{aligned} &- 7F_n^9 - 477F_{n+1}^9 + 12519F_{n+2}^9 + 204516F_{n+3}^9 + 165100F_{n+4}^9 - 204516F_{n+5}^9 \\ &+ 12519F_{n+6}^9 + 477F_{n+7}^9 - 7F_{n+8}^9 = 262080F_{3n+12}^3. \end{aligned} \tag{3.5}$$

For $(p, m, k) = (3, 2, 2)$ another instance of Conjecture 3.2 is

$$2F_n^6 - 803F_{n+2}^6 + 34034F_{n+4}^6 - 34034F_{n+6}^6 + 803F_{n+8}^6 - 2F_{n+10}^6 = 27720F_{2n+10}^3. \tag{3.6}$$

We invite the reader to check the validity of (3.3)-(3.6), and to also check that in each case the stated symmetry conditions on the a_i are satisfied.

In fact, identity (2.4) is a family of identities with $(p, m) = (2, 2)$, and each identity in this family is an instance of Conjecture 3.1. Likewise, (2.5) is a family of identities in which each identity is an instance of Conjecture 3.2, and (2.6) is a family of identities in which each identity is an instance of Conjecture 3.1.

4. A Sample Proof

Each result in this paper can be proved with the use of a method introduced by Dresel [1]. To illustrate, we prove Theorem 2.1.

In the terminology of Dresel, (2.4) is homogeneous of degree 4 in the variable n. Next we look at the variable k. As Dresel explains, since $(-1)^k = (\alpha\beta)^k$, where α and β are the roots of $x^2 - x - 1 = 0$, then $(-1)^k$ is of degree 2 in the variable k. Accordingly, into certain terms of (2.4) we insert appropriate powers of $(-1)^k$ in order to make (2.4) homogeneous of degree 19 in the variable k. For instance, we write the first terms on the left as $(-1)^{9k+1}F_kF_n^4$, and the middle term on the left as $-F_{3k}\left((-1)^{2k}L_{4k} + 2(-1)^{3k}L_{2k} + 4(-1)^{4k}\right)F_{n+2k}^4$.

We noted above that (2.4) is homogeneous of degree 4 in the variable n. Therefore, to prove (2.4) with the Verification Theorem of Dresel [1, page 171], we need only verify its validity for five distinct values of n. Accordingly, we write down the cases that correspond to $n = 1, 2, 3, 4$, and 5. We are required to prove each of these five cases. Now, each of these five cases is an identity that is homogeneous of degree 19 in the variable k. Therefore, to prove any one of these five cases, we need only verify its validity for twenty distinct values of k; say $k = 1, 2, \ldots, 20$. We are required to verify (2.4) for 5×20 distinct ordered pairs (n, k). We managed to perform these verifications and thereby complete the proof of Theorem 2.1 in a matter of seconds with the use of the computer algebra system *Mathematica* 6.0.

5. Certain Expansions in Terms of Lucas Numbers

Our discovery of (2.4)-(2.6) became routine when we realized that factors of some coefficients could be expanded in terms of Lucas numbers with even subscripts. This prompted us to search for Fibonacci/Lucas expressions that have such expansions. The most interesting such expansions that we found are the expansions of L_n^{2m} and F_n^{2m}. To establish these expansions, we require the three preliminary results that follow.

$$5F_n^2 = L_n^2 + 4(-1)^{n+1}, \tag{5.1}$$

$$\sum_{k=0}^{m} (-1)^k 4^{m-k} \binom{m}{k}\binom{2k}{k} = \binom{2m}{m}, \tag{5.2}$$

and

$$\sum_{k=i}^{m} (-1)^k 4^{m-k} \binom{m}{k}\binom{2k}{k-i} = (-1)^i \binom{2m}{m-i}. \tag{5.3}$$

Identity (5.1) occurs as I_{12} on page 56 of [3]. Identity (5.2) occurs as (3.85) in [2]. Furthermore, identity (5.2) is proved by two different methods in [5], see pages 116 and 123. Identity (5.3) is a generalization of identity (5.2). However, since we have not cited (5.3) in the literature available to us, we present a short proof.

To prove (5.3) we use the method of WZ pairs as described in Wilf [5, pp 120-126]. In the terminology of Wilf, identity (5.3) is certified by the rational function

$$R(m, k) = \frac{2k - 1}{2m + 1},$$

a fact that can be verified with Gosper's algorithm.

We designate the expansion of L_n^{2m} as a lemma, and the expansion of F_n^{2m} as a theorem.

Lemma 5.1. *Let m be a positive integer. Then*

$$L_n^{2m} = \sum_{i=0}^{m} \binom{2m}{m-i} (-1)^{(m-i)n} L_{2in} + (-1)^{mn+1} \binom{2m}{m}. \tag{5.4}$$

The proof of Lemma 5.1 is immediate if we take the binet form of L_n and expand L_n^{2m}. We leave the details to the reader.

Theorem 5.2. *Let m be a positive integer. Then*

$$5^m F_n^{2m} = \sum_{i=1}^{m} (-1)^{(m+i)(n+1)} \binom{2m}{m-i} L_{2in} + (-1)^{m(n+1)} \binom{2m}{m}. \tag{5.5}$$

Proof. From (5.1) and the binomial theorem we obtain

$$5^m F_n^{2m} = \sum_{k=0}^{m} \binom{m}{k} L_n^{2k} (-1)^{(n+1)(m-k)} 4^{m-k}. \tag{5.6}$$

In (5.6) we substitute the right side of (5.4) for L_n^{2k} to obtain a double sum. In this double sum we reverse the order of summation to obtain

$$\begin{aligned} 5^m F_n^{2m} = (-1)^{m(n+1)} \Bigg(\sum_{i=1}^{m} (-1)^{in} L_{2in} \sum_{k=i}^{m} (-1)^k 4^{m-k} \binom{m}{k} \binom{2k}{k-i} \\ + \sum_{k=0}^{m} (-1)^k 4^{m-k} \binom{m}{k} \binom{2k}{k} \Bigg). \end{aligned} \tag{5.7}$$

Theorem 5.2 now follows from (5.2) and (5.3). □

References

[1] L. A. G. Dresel, *Transformations of Fibonacci-Lucas identities*, Applications of Fibonacci Numbers, **5** (1993), 169–184.

[2] H. W. Gould, *Combinatorial Identities*, Morgantown, West Virginia, 1972.

[3] Verner E. Hoggatt, Jr, *Fibonacci and Lucas Numbers*, Boston:Houghton-Mifflin, 1969. Reprinted, The Fibonacci Association, 1972.

[4] R. S. Melham, *On certain combinations of higher powers of Fibonacci numbers*, The Fibonacci Quarterly, **48.3** (2010) 256–259.

[5] Herbert S. Wilf, *generatingfunctionology*, Academic Press, 1990.

MSC2000: 11B39.

Department of Mathematical Sciences, University of Technology, Sydney, PO Box 123, Broadway, NSW 2007 Australia

E-mail address: ray.melham@uts.edu.au