FAMILIES OF FIBONACCI AND LUCAS SUMS VIA THE MOMENTS OF A RANDOM VARIABLE

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Abstract. In this paper we show how the expectation of a particular random variable gives rise to an infinite series whose coefficients are certain functions of the Fibonacci numbers. A general result follows from this which, when specialized to varying degrees, leads both to well-known and lesser-known identities. Also, by considering a different random variable, we go on to obtain corresponding results for the Lucas numbers. Finally, we look at series arising from higher moments.

1. Introduction

In probability theory it is frequently the case that the random variables of interest are themselves functions of other, generally simpler, random variables. As an example of this, consider the random variable $X \sim B(n, p)$. This denotes the fact that X follows a binomial distribution for which the number of trials is n and the probability of success on a particular trial is equal to p. Note that we may regard X as the sum of n Bernoulli random variables, each identically and independently distributed as the random variable B possessing the mass function $f(b)$ given by $f(0) = 1 - p$ and $f(1) = p$, where $0 < p < 1$.

The random variables Y and W we consider here are a little more complicated than X, yet are nonetheless functions of a series of identically and independently distributed random variables. We look at the infinite series that arise as a consequence of taking the limits of the expectations of Y and W, and arrive at general results concerning series involving Fibonacci and Lucas numbers. On specializing these results we are able to generate a number of well-known identities in addition to some that may be less familiar.

We first set up the machinery for defining Y. Let B be the Bernoulli random variable referred to in the previous paragraph, and let C be a 'constant' random variable taking the value 1, with mass function $g(c)$ given by $g(1) = 1$. Next, let $\{k_1, k_2, k_3, \ldots\}$ be a strictly increasing sequence of positive integers. Finally, we define $\{X_1, X_2, X_3, \ldots\}$ to be a sequence of independent random variables for which X_n is distributed as B if $n = k_m$ for some $m \in \mathbb{N}$, but as C otherwise.

Let us now consider, for a particular sequence $\{k_1, k_2, k_3, \ldots\}$, the random variable Y defined by way of the following infinite continued fraction:

$$Y = \cfrac{X_1}{1 + \cfrac{X_2}{1 + \cfrac{X_3}{1 + \cdots}}}.$$

(See [1] and [5] for details of mathematical properties of continued fractions.) First, if $X_1 = 0$ then $Y = 0$. Next, suppose that for some $k_r \geq 2$, $X_{k_r} = 0$ but $X_m = 1$ for $m = 1, 2, \ldots, k_r - 1$. Then Y is the reciprocal of the $(k_r - 1)$th convergent to $\phi = \frac{1+\sqrt{5}}{2}$, the *golden ratio*. Thus in this case Y is given by

$$Y = \frac{F_{k_r - 1}}{F_{k_r}}.$$

We next calculate $E(Y)$, the expectation of Y. To this end,

$$\begin{aligned}
\mathrm{P}\left(Y = \frac{F_{k_r-1}}{F_{k_r}}\right) &= \mathrm{P}(X_1 = X_2 = \ldots = X_{k_r - 1} = 1 \quad \text{and} \quad X_{k_r} = 0) \\
&= p^{r-1}(1-p).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{E}(Y) &= \sum_{r=1}^{\infty} \mathrm{P}\left(Y = \frac{F_{k_r-1}}{F_{k_r}}\right) \frac{F_{k_r-1}}{F_{k_r}} \\
&= \sum_{r=1}^{\infty} p^{r-1}(1-p)\frac{F_{k_r-1}}{F_{k_r}} \\
&= \frac{F_{k_1-1}}{F_{k_1}} - p\frac{F_{k_1-1}}{F_{k_1}} + p\frac{F_{k_2-1}}{F_{k_2}} - p^2\frac{F_{k_2-1}}{F_{k_2}} + \cdots . \qquad (1.1)
\end{aligned}$$

On noting that

$$\frac{F_{k_1-1}}{F_{k_1}} + p\frac{F_{k_1-1}}{F_{k_1}} + p\frac{F_{k_2-1}}{F_{k_2}} + p^2\frac{F_{k_2-1}}{F_{k_2}} + \cdots$$

is a convergent series (see [4], remembering that $0 < p < 1$), it follows that (1.1) is an absolutely convergent series. Thus, we may rearrange its terms without altering the sum to obtain

$$\begin{aligned}
\mathrm{E}(Y) &= \frac{F_{k_1-1}}{F_{k_1}} + p\left(\frac{F_{k_2-1}}{F_{k_2}} - \frac{F_{k_1-1}}{F_{k_1}}\right) + p^2\left(\frac{F_{k_3-1}}{F_{k_3}} - \frac{F_{k_2-1}}{F_{k_2}}\right) + \cdots \\
&= \frac{F_{k_1-1}}{F_{k_1}} + \sum_{r=1}^{\infty} p^r \left(\frac{F_{k_{r+1}-1}}{F_{k_{r+1}}} - \frac{F_{k_r-1}}{F_{k_r}}\right).
\end{aligned}$$

Then, using d'Ocagne's identity $F_{n+1}F_m - F_nF_{m+1} = (-1)^n F_{m-n}$, which may be found in [7], it follows that

$$\begin{aligned}
\mathrm{E}(Y) &= \frac{F_{k_1-1}}{F_{k_1}} + \sum_{r=1}^{\infty} p^r \frac{F_{k_r}F_{k_{r+1}-1} - F_{k_r-1}F_{k_{r+1}}}{F_{k_r}F_{k_{r+1}}} \\
&= \frac{F_{k_1-1}}{F_{k_1}} + \sum_{r=1}^{\infty} p^r \frac{(-1)^{k_r-1}F_{k_{r+1}-k_r}}{F_{k_r}F_{k_{r+1}}}.
\end{aligned}$$

Also

$$\begin{aligned}
\left|\frac{(-1)^{k_r-1}F_{k_{r+1}-k_r}}{F_{k_r}F_{k_{r+1}}}\right| &= \frac{F_{k_{r+1}-k_r}}{F_{k_r}F_{k_{r+1}}} \\
&\leq \frac{1}{F_{k_r}},
\end{aligned}$$

so the sum

$$\sum_{r=1}^{\infty} \frac{(-1)^{k_r-1}F_{k_{r+1}-k_r}}{F_{k_r}F_{k_{r+1}}}$$

does indeed exist. Thus, by Abel's Theorem (see [3] or [8]), it is the case that

$$\lim_{p\to 1^-}\sum_{r=1}^{\infty} p^r \frac{(-1)^{k_r-1}F_{k_{r+1}-k_r}}{F_{k_r}F_{k_{r+1}}} = \sum_{r=1}^{\infty}\frac{(-1)^{k_r-1}F_{k_{r+1}-k_r}}{F_{k_r}F_{k_{r+1}}}.$$

Then, noting that $\lim_{p\to 1^-}\mathrm{E}(Y) = \frac{1}{\phi}$, we obtain the result

$$\frac{F_{k_1-1}}{F_{k_1}} + \sum_{r=1}^{\infty}\frac{(-1)^{k_r-1}F_{k_{r+1}-k_r}}{F_{k_r}F_{k_{r+1}}} = \frac{1}{\phi} = \frac{\sqrt{5}-1}{2}, \tag{1.2}$$

where $\lim_{p\to 1^-}$ denotes the limit as p approaches 1 from below.

2. Special Cases

On letting $k_n = n$ in (1.2) we have the well-known result

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{F_rF_{r+1}} = \frac{1}{\phi}, \tag{2.1}$$

which can be found on the website [2] as result (79). However, the following generalization of (2.1), obtained from (1.2) by setting $k_n = m(n-1)+1$ for some $m \in \mathbb{N}$, might not be quite so well-known:

$$\begin{aligned}\frac{1}{\phi} &= \frac{F_0}{F_1} + \sum_{r=1}^{\infty}\frac{(-1)^{m(r-1)}F_{(mr+1)-(m(r-1)+1)}}{F_{m(r-1)+1}F_{mr+1}}\\ &= \sum_{r=1}^{\infty}\frac{(-1)^{m(r-1)}F_m}{F_{m(r-1)+1}F_{mr+1}}.\end{aligned}$$

When m is even, $m = 2n$ say, we have the particularly appealing formula

$$\sum_{r=1}^{\infty}\frac{1}{F_{2n(r-1)+1}F_{2nr+1}} = \frac{1}{\phi F_{2n}}.$$

Also, with $k_n = n^2$, we obtain

$$\begin{aligned}\frac{1}{\phi} &= \sum_{r=1}^{\infty}\frac{(-1)^{r^2-1}F_{(r+1)^2-r^2}}{F_{r^2}F_{(r+1)^2}}\\ &= \sum_{r=1}^{\infty}\frac{(-1)^{r-1}F_{2r+1}}{F_{r^2}F_{(r+1)^2}},\end{aligned}$$

and indeed similar results follow for any strictly increasing polynomial sequences.

Next, with $k_n = 2^{n-1}$, it follows that

$$\begin{aligned}\frac{1}{\phi} &= \sum_{r=1}^{\infty}\frac{(-1)^{2^{r-1}-1}F_{2^r-2^{r-1}}}{F_{2^{r-1}}F_{2^r}}\\ &= \sum_{r=1}^{\infty}\frac{(-1)^{2^{r-1}-1}}{F_{2^r}}\\ &= 1 - \sum_{r=2}^{\infty}\frac{1}{F_{2^r}},\end{aligned}$$

giving

$$\sum_{r=0}^{\infty} \frac{1}{F_{2^r}} = \frac{1}{F_1} + \frac{1}{F_2} + \sum_{r=2}^{\infty} \frac{1}{F_{2^r}}$$
$$= 2 + \frac{3-\sqrt{5}}{2}$$
$$= \frac{7-\sqrt{5}}{2},$$

which is known as the *Millin series* [6].

Taking this idea further, we have, with $k_n = 3^{n-1}$,

$$\frac{1}{\phi} = \sum_{r=1}^{\infty} \frac{(-1)^{3^{r-1}-1} F_{3^r - 3^{r-1}}}{F_{3^{r-1}} F_{3^r}}$$
$$= \sum_{r=1}^{\infty} \frac{F_{2 \cdot 3^{r-1}}}{F_{3^{r-1}} F_{3^r}}.$$

Then, using the results $F_{2r} = F_r \left(F_{r-1} + F_{r+1}\right)$ and $L_r = F_{r-1} + F_{r+1}$, we obtain the remarkable identity

$$\frac{1}{\phi} = \sum_{r=0}^{\infty} \frac{F_{3^r - 1} + F_{3^r + 1}}{F_{3^{r+1}}}$$
$$= \sum_{r=0}^{\infty} \frac{L_{3^r}}{F_{3^{r+1}}}.$$

Finally, we may index the Fibonacci numbers with themselves to give the rather amusing result

$$\sum_{r=1}^{\infty} \frac{(-1)^{F_{r+1}-1} F_{F_r}}{F_{F_{r+1}} F_{F_{r+2}}} = \frac{1}{\phi}.$$

3. Lucas Sums

Now let B_1 and B_2 be a pair of jointly-distributed discrete random variables with joint mass function $f(b_1, b_2)$ given by $f(3,0) = 1 - p$ and $f(1,1) = p$, where $0 < p < 1$, and let C_1 and C_2 be a pair of 'constant' random variables with joint mass function $g(c_1, c_2)$ given by $g(1,1) = 1$. Once more $\{k_1, k_2, k_3, \ldots\}$ represents a strictly increasing sequence of positive integers, although we now impose the extra condition that $k_1 \geq 2$. We define $\{(U_2, V_2), (U_3, V_3), (U_4, V_4), \ldots\}$ to be a sequence of independent pairs of random variables for which (U_n, V_n) is distributed as (B_1, B_2) if $n = k_m$ for some $m \in \mathbb{N}$, but as (C_1, C_2) otherwise (the fact that the indexing starts at 2 is simply a matter of notational convenience).

We now consider the random variable W given by

$$W = \cfrac{1}{U_2 + \cfrac{V_2}{U_3 + \cfrac{V_3}{U_4 + \cdots}}}.$$

Arguing as in Section 1, we obtain

$$P\left(W=\frac{L_{k_r-1}}{L_{k_r}}\right)=P((U_{k_i},V_{k_i})=(1,1)\quad\text{for}\quad i=1,2,\ldots,r-1\quad\text{and}\quad(U_{k_r},V_{k_r})=(3,0))$$
$$=p^{r-1}(1-p),$$

which leads eventually to the result

$$\lim_{p\to 1^-}\mathrm{E}(W)=\frac{1}{\phi}=\frac{L_{k_1-1}}{L_{k_1}}+\sum_{r=1}^{\infty}\frac{L_{k_r}L_{k_{r+1}-1}-L_{k_r-1}L_{k_{r+1}}}{L_{k_r}L_{k_{r+1}}}. \tag{3.1}$$

There is no equivalent of d'Ocagne's identity for the Lucas numbers, so the numerator under the sum in (3.1) cannot be simplified in general. However, we do have the result $L_n^2-L_{n-1}L_{n+1}=5(-1)^n$. Thus, with $k_n=n+1$, $n=1,2,3,\ldots$, it follows that

$$\frac{1}{\phi}=\frac{L_1}{L_2}+\sum_{r=1}^{\infty}\frac{L_{r+1}^2-L_rL_{r+2}}{L_{r+1}L_{r+2}}$$
$$=\frac{1}{3}+\sum_{r=1}^{\infty}\frac{5(-1)^{r+1}}{L_{r+1}L_{r+2}},$$

which gives

$$\sum_{r=1}^{\infty}\frac{5(-1)^{r-1}}{L_rL_{r+1}}=\frac{5}{L_1L_2}-\sum_{r=1}^{\infty}\frac{5(-1)^r}{L_{r+1}L_{r+2}}$$
$$=\frac{5}{3}-\left(\frac{1}{\phi}-\frac{1}{3}\right).$$

On simplifying, we arrive at the result

$$\sum_{r=1}^{\infty}\frac{(-1)^{r-1}}{L_rL_{r+1}}=\frac{3-\phi}{5}.$$

4. Higher Moments

Using similar arguments to those given previously, we see that

$$\lim_{p\to 1^-}\mathrm{E}(Y^n)=\frac{1}{\phi^n}=\left(\frac{F_{k_1-1}}{F_{k_1}}\right)^n+\sum_{r=1}^{\infty}\frac{\left(F_{k_r}F_{k_{r+1}-1}\right)^n-\left(F_{k_r-1}F_{k_{r+1}}\right)^n}{\left(F_{k_r}F_{k_{r+1}}\right)^n}. \tag{4.1}$$

For the special case $n=2$ and $k_r=r$, for $r=1,2,3,\ldots$, it follows from (4.1) that

$$\frac{1}{\phi^2}=\sum_{r=1}^{\infty}\frac{F_r^4-(F_{r-1}F_{r+1})^2}{(F_rF_{r+1})^2}$$
$$=\sum_{r=1}^{\infty}\frac{\left(F_r^2-F_{r-1}F_{r+1}\right)\left(F_r^2+F_{r-1}F_{r+1}\right)}{(F_rF_{r+1})^2}$$
$$=\sum_{r=1}^{\infty}\frac{(-1)^{r-1}\left(2F_r^2+(-1)^r\right)}{(F_rF_{r+1})^2}.$$

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