ALMOST-RECURSIVENESS OF RECIPROCALS OF LINEARLY RECURRENT SEQUENCES

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ABSTRACT. The paper provides an approximation up to a bounded additive error for the sequence whose general term is the reciprocal of the tail of series of reciprocals of a given binary recurrent sequence.

1. INTRODUCTION AND GOALS

Let $\{G_n\}_{n\geq 0}$ be a linear recurrence satisfying the recursion

$$G_n = \sum_{i=1}^{m} a_i G_{n-i}, \quad n \geq m, \tag{1.1}$$

with $a_i \in \mathbb{R}, 1 \leq i \leq m, m \in \mathbb{N}$ and with $G_n > 0$, for $n \geq 1$. For integer $n \geq 1$, we define the *associated* sequences

$$h_n = h_n\left(\{G_t\}_{t\geq 0}\right) = \frac{1}{\sum_{i=n}^{\infty} \frac{1}{G_i}}, H_n = H_n\left(\{G_t\}_{t\geq 0}\right) = \lfloor h_n + .5 \rfloor. \tag{1.2}$$

Let $\{G'_n\}_{n\geq 0}$ be another linear recurrence satisfying the recursion

$$G'_n = \sum_{i=1}^{m'} a'_i G'_{n-i}, \quad n \geq m, \tag{1.3}$$

with $a'_i \in \mathbb{R}, 1 \leq i \leq m', m' \in \mathbb{N}$. We define the *test sequence of* $\{G_n\}_{n\geq 0}$ *relative to the recursion* (1.3) by

$$t_n = H_n - \sum_{i=1}^{m'} a'_i H_{n-i}, \quad n \geq m' + 1. \tag{1.4}$$

We say that the sequence $\{G_n\}_{n\geq 0}$ is *almost-recursive with respect to the recursion* (1.3) if $t_n = O(1)$. If in (1.1) and (1.3) $m = m'$, and $a_i = a'_i, 1 \leq i \leq m$, we say that the sequence $\{G_n\}_{n\geq 0}$ is *almost-recursive with respect to itself*, or simply *almost-recursive.*

Ohtsuka and Nakamura [1] prove that for integer $n \geq 2$,

$$\left\lfloor \frac{1}{\sum_{i=n}^{\infty} \frac{1}{F_i}} \right\rfloor = F_{n-2} + \left(\frac{(-1)^n - 1}{2}\right). \tag{1.5}$$

The corresponding identity when the nearest integer function is used is

$$H_n\left(\{F_m\}_{m\geq 0}\right) = F_{n-2}, \quad n \geq 2. \tag{1.6}$$

While revising this paper, the author learned that Komatsu [2] proved a variety of generalizations of (1.5). For example, he proves that if $u_n = au_{n-1} + u_{n-2} + \cdots + u_{n-s}, n \geq s, u_0 \geq 0, u_k \in \mathbb{N}, 0 \leq k \leq s-1, a, s \in N$ then there is a constant n_0 such that

$$H_n\left(\{u_m\}_{m\geq 0}\right) = u_n - u_{n-1}, \quad n \geq n_0. \tag{1.7}$$

Notice that (1.7) includes as special cases the Pell numbers and the Generalized Fibonacci numbers with recursion $G_n = aG_{n-1} + G_{n-2}, G_0 = 0, G_1 = 1$. We can reformulate (1.6) and (1.7) as stating that the Fibonacci numbers and the sequence $\{u_n\}_{n\geq 0}$ are almost-recursive with respect to themselves with their test sequences eventually becoming identically zero.

We can now outline the rest of this paper. Given an arbitrary second order recursion,

$$G_n = ca^n + db^n, \text{ with } a, b, c, d \in \mathbb{R}, cd \neq -1, a > 0, \max\{1, |b|\} < a, \tag{1.8}$$

the main theorem, proven in Section 3, with the consequences to it proven in Section 4, explicitly computes a sequence $\{G'_n\}_{n\geq 0}$ such that $\{G_n\}_{n\geq 0}$ is almost recursive with respect to any recursion which $\{G'_n\}_{n\geq 0}$ satisfies. We are then assured that the test sequence is bounded.

However the main theorem says nothing about the nature of the test sequence (other than it is bounded). The test sequences of recursions that are almost recursive with respect to another sequence do not seem to have any special properties. However, the test sequences of recursions that are almost recursive with respect to themselves exhibit periodicity and a variety of other interesting properties. Accordingly, in Section 2, we explore many examples and present several conjectures and open problems.

2. Examples and Conjectures

Throughout this section we study sequences $\{G_n\}_{n\geq 0}$ satisfying (1.8) and also satisfying

$$b^2 < a. \tag{2.1}$$

Knowledge of the full statement of the main theorem and its corollaries is not needed in this section. It suffices to cite two results: (i) Corollary 4.3 of Section 4 which asserts that any recursive sequence satisfying (1.8) and (2.1) is almost recursive with respect to itself, and (ii) Proposition 4.1 which implies that $\{G_n\}_{n\geq 0}$ satisfies the recursion $G_n = (a+b)G_{n-1} - abG_{n-2}$. Using (1.2) and (1.4) we can then compute the test sequence, $\{t_n\}_{n\geq 3}$.

We first introduce well-known terminology and notation useful in describing these sequences.

Definition 2.1. *A sequence is* periodic *if for some non-negative integer l, $t_n = t_{n+l}$ for all $n > n_0$. Without loss of generality, we may assume l and n_0 smallest.*

l is called the length *of the period. Furthermore, if $n_0 = 0$ then the sequence is called purely periodic; otherwise it is called* periodic*, or if we wish to give emphasis,* eventually periodic*.*

We notationally indicate the period *by $\langle p_i, i = 1, \ldots, l\rangle$.*

We follow the usual convention of letting $p_{i+ml} = p_i, m \in N$.

We say the period is anti-symmetric *if l is even and if $p_i = p_{i+l/2}$, $1 \leq i \leq l/2$.*

Two periods, p, q, of two different sequences are said to be similar *if their lengths are equal and for some non-negative integer, c, $p_i = \varepsilon q_{i+c}$, $1 \leq i \leq l$, $\varepsilon \in \{-1, 1\}$. Intuitively, similarity refers to a translation with a possible sign change. Since similarity is an equivalence relation, we may freely speak about the* period class *or the* similarity class*.*

Prior to presenting the next definition we first mention conjectures which hold in all examples studied.

Conjecture 2.2. *If* (1.8) *and* (2.1) *hold then the test sequence is eventually periodic.*

Comment: Eventual periodicity also seems to hold for sequences of higher orders, for example the Tribonacci numbers (cf. [2]), provided these sequences are almost recursive with respect to themselves. However, we have not found periodicity in sequences which are almost recursive with respect to other sequences, even though the main theorem and its consequences state that their test sequences are bounded.

Conjecture 2.3. *Suppose* (1.8) *and* (2.1) *hold. Suppose further that the test sequence is eventually periodic with period p and (period) length l. Then if p is not identically zero, then* $\max_{1\leq i\leq l} p_i > 0$ *and* $\min_{1\leq i\leq l} p_i < 0$.

Comment: Note, Conjecture 2.3 is true in all examples reviewed, independent of the signs of c and d. Note, that if p and p' are two similar periods that differ by a sign then their max and min will be reversed. This motivates the next definition.

Definition 2.4. *The* max-min *set of a period is* $(\max|p_i|, \min|p_i|)$.

We now present examples. We start with a simple example illustrating many of our previous definitions. In the remainder of the paper we use parenthesis, braces, and angle-brackets to indicate sets, infinite sequences, and finite sequences, respectively.

Example 2.5. *In* (1.8), *let* $\langle a, b, c, d\rangle = \langle 18, 4, 1, 1\rangle$. *Using* (1.2), *we can compute*

$$H_n\big(\{G_t\}_{t\geq 0}\big)_{n\geq 1} = \{21, 320, 5566, 99375, 1785517, 32126355, \ldots\}.$$

By (1.8), (2.1), *and Conjecture 2.2, the sequence* $\{G_n\}_{n\geq 0}$, *is almost recursive with respect to itself. By* (1.4), $\{t_n\}_{n\geq 3} = \{38, -37, 19, -19, 19, -19, 19, -19, \ldots\}$. *Thus the sequence* $\{t_n\}_{n\geq 3}$ *appears to be eventually periodic with period length 2 with the underlying period possessing anti-symmetry.*

The following example gives an interesting illustration of the known fact that an assertion of periodicity is dependent on the length of the initial segment of the sequence computed.

Example 2.6. *In* (1.8), *let* $a = 17$, $b = 4$, $c = -1$, $d = -3$. *Define vectors*

$$v_1 = \langle -17, -2, 35, \mathbf{-18}, \mathbf{15}, \mathbf{32}, \mathbf{33}, 17, \mathbf{-1}, \mathbf{15}, \mathbf{34}, -1, 17, 0, -1, 18, -16, -16\rangle,$$

$$v_2 = \langle -17, -2, 35, \mathbf{-19}, \mathbf{35}, \mathbf{-15}, \mathbf{-35}, 17, \mathbf{-1}, \mathbf{15}, \mathbf{34}, -1, 17, 0, -1, 18, -16, -16\rangle, \text{ and}$$

$$v_3 = \langle -17, -2, 35, \mathbf{-19}, \mathbf{35}, \mathbf{-15}, \mathbf{-35}, 17, \mathbf{-2}, \mathbf{36}, \mathbf{-34}, -1, 17, 0, -1, 18, -16, -16\rangle.$$

In defining the $v_i, i = 1, 2, 3$, we have used boldface to highlight the differences between the v_i. More specifically, v_1 differs from v_2 at positions 4,5,6,7 and v_2 and v_3 differ at positions 9,10, and 11. By (1.4), $\langle t_1, t_2, \ldots, t_{198}\rangle = v_1v_2v_2v_3v_3v_3v_3v_3v_3v_3v_3$.

If we had only computed 54 terms then the sequence $v_1v_2v_2$ suggests that v_2 is the period. However, after computing 198 terms we see that v_3 is probably the true period.

This example emphasizes the fact that each assertion of periodicity in the examples presented in this section is a conjecture. Any particular example can be proven using either the algebraic methods of Ohtsuka and Nakamura [1], or the analytic (power series) methods of Komatsu [2]. The challenge in these conjectures and open problems is to find proofs that cover a wide range of cases.

In studying further examples we focus on the following five attributes of test sequences and their periods.

* The period, (actually a representative of the period class), p,
* The period length, l,

* The max-min set, $(\max_{1\le i\le l}|p_i|, \min_{1\le i\le l}|p_i|)$,
* The period sum, $\sum_{1\le i\le l} p_i$, and
* Anti-symmetry.

However, in the examples we reviewed, pure periodicity was rare, and therefore, it is not studied further in this section.

a	18	22	34	38	42	46	62
l	2	2	1	6	10	5	2
$max - min$	(19,19)	(23,23)	(0,0)	(78,76)	(85,44)	(93,47)	(62,1)
$\sum p_i$	0	0	0	37	0	-135	-61

Table 1. Period lengths, max-min sets, and period sums for the given value of a with $b = 4$, $c = 1$, $d = 1$ in (1.8).

Table 1 shows these attributes for $\langle a, b, c, d\rangle = \langle 18 + 4i, 4, 1, 1\rangle, i \in \{0, 1, 4, 5, 6, 7, 11\}$. In interpreting the table notice that we have anti-symmetry when $a = 18, 22$. To see this, recall by the conjectures that the max and min have opposite signs. Since the period length is 2, it follows that $\langle 19, -19\rangle$ and $\langle 23, -23\rangle$ are representatives of the similarity class of the periods of the test sequences for $a = 18$ and $a = 22$, respectively. A more sophisticated example of anti-symmetry occurs when $\langle a, b, c, d\rangle = \langle 70, 4, 1, 1\rangle$ with period $p = \langle 70, 1, -70, -1\rangle$.

In Table 1, notice that certain periods are anti-symmetric, certain periods have singleton max-min sets, and certain periods have zero period-sums (even though they are not anti-symmetric and even though their max and min differ). Table 1, as well as the other examples and table in this section, suggest obvious open questions.

Open Questions:

* Are there an infinite number of quadruples $\langle a, b, c, d\rangle$ whose periods have a given period length?
* For which quadruples and how frequently do we have $\sum p_i = 0$?
* Can we characterize those quadruples, and describe the frequency of quadruples, with antisymmetric periods or with singleton max-min sets?
* These periods of test sequences are reminiscent of the periodic continued fractions of quadratic irrationals. The analogy suggests seeking parametrically defined infinite subsets of quadruples whose periods exhibit common patterns.

(a, b)	(10,3)	(13,3)	(16,3)	(17,4)	(24,4)	(34,5)	(39,6)
l	48	41	55	18	30	575	10
$max - min$	(11,11)	(14,14)	(17,17)	(35,35)	(49,50)	(70,70)	(120,118)
$\sum p_i$	0	72	0	0	69	0	0

Table 2. Period lengths, max-min sets, and period sums for the given value of a, b with $c = 1$, $d = 1$, in (1.8). The only example with anti-symmetry is $a = 10$, $b = 3$.

Table 2 explores several other quadruples with a variety of properties. Again the table suggests several open questions.

We close this section with an example exploring certain parameters varying while others remain fixed.

Example 2.7. *In* (1.8) *let* $a = 5$ *and* $b = 2$. *As we let* c, d *vary over a wide range of integer pairs we observe that the periods of the associated test sequences are always similar to* $\langle 0, 0, 1, -6, 6, -5, -1, 5, 1, -6, 5\rangle$.

Several other pairs of $\langle a, b\rangle$ with small b seem to have one period class independent of the choice of $\langle c, d\rangle$. In the general case although we don't have single period classes the period lengths seem to have a non-trivial common multiple. The number of period lengths associated with a fixed a and b also appears to sometimes be bounded (as c and d arbitrarily vary).

3. The Main Theorem

In this section we present and prove the main theorem.

Theorem 3.1. *Let* a, b, c, d *and* $\{G_n\}_{n\geq 0}$ *be defined as in* (1.8). *Let* k_0 *be the unique non-negative integer satisfying,*

$$\left|\frac{b^{k_0+1}}{a^{k_0}}\right| < 1 \leq \left|\frac{b^{k_0}}{a^{k_0-1}}\right|. \tag{3.1}$$

Let h_n *be defined by* (1.2). *Then for sufficiently large* n, *there are computable constants* $e_k, k = 1, 2, \dots,$ *such that*

$$h_n = h_n\left(\{G_t\}_{t\geq 0}\right) = c\frac{a-1}{a}a^n + c\frac{a-1}{a}\sum_{k=1}^{k_0} e_k\left(\frac{b^k}{a^{k-1}}\right)^n + O(1). \tag{3.2}$$

Proof. By (1.8),

$$G_m = ca^m\left(1 + \frac{db^m}{ca^m}\right).$$

Also by (1.8), for sufficiently large m, the rightmost fraction above is bounded by 1 in absolute value. Thus, for sufficiently large m,

$$\begin{aligned}
\frac{1}{G_m} &= \frac{1}{ca^m}\left(1 + \frac{db^m}{ca^m}\right)^{-1}\\
&= \frac{1}{ca^m}\left(1 + \sum_{k\geq 1}(-1)^k\left(\frac{d}{c}\right)^k\left(\frac{b}{a}\right)^{mk}\right)\\
&= \frac{1}{ca^m} + \sum_{k\geq 1}(-1)^k\left(\frac{d^k}{c^{k+1}}\right)\left(\frac{b^k}{a^{k+1}}\right)^m.
\end{aligned}$$

Taking sums and regrouping to emphasize the corresponding geometric series, we obtain, using (1.2), that for sufficiently large n,

$$\begin{aligned}\frac{1}{h_n} &= \sum_{m\geq n}\frac{1}{G_m}\\ &= \frac{1}{c\frac{a-1}{a}a^n}+\sum_{k\geq 1}(-1)^k\left(\frac{d^k}{c^{k+1}}\right)\left(\frac{a^{k+1}}{a^{k+1}-b^k}\right)\left(\frac{b^k}{a^{k+1}}\right)^n \\ &= \frac{1}{c\frac{a-1}{a}a^n}\left(1+\sum_{k\geq 1}f_k\left(\frac{b}{a}\right)^{kn}\right),\end{aligned} \tag{3.3}$$

where

$$f_k=(-1)^k\frac{d^k}{c^k}\frac{a^k(a-1)}{a^{k+1}-b^k}. \tag{3.4}$$

In taking these sums, we need not worry about absolute convergence since we are working with exponentially decaying sequences. By considering the two cases of positive or negative b in the equation $f_k=(-1)^k\left(\frac{d}{c}\right)^k\frac{a-1}{a-(\frac{b}{a})^k}$, and by (1.8), we see

$$|f_k|<\left(\left|\frac{d}{c}\right|\right)^k,\quad k\geq 1. \tag{3.5}$$

Consequently, it follows that for

$$L>\frac{\log(|\frac{d}{c}|)}{\log\left(\frac{a}{|b|}\right)},$$

and for sufficiently large N, that for all $n>N>L$,

$$\left|\sum_{k\geq 1}f_k\left(\frac{b}{a}\right)^{kn}\right|<\sum_{k\geq 1}\left(\left|\frac{d}{c}\right|\right)^k\left(\frac{|b|}{a}\right)^{kn}\leq\sum_{k\geq 1}\left(\frac{|b|}{a}\right)^{k(N-L)}<\left(\frac{|b|}{a}\right)^{(N-L)}<1. \tag{3.6}$$

Define

$$Z=\sum_{k\geq 1}f_k\left(\frac{b}{a}\right)^{kn}. \tag{3.7}$$

Then (3.6) shows $|Z|=\left|\sum_{k\geq 1}f_k\left(\frac{b}{a}\right)^{kn}\right|<1$. Hence, by (3.3) we obtain

$$h_n=c\frac{a-1}{a}a^n\left(1+Z\right)^{-1}=c\frac{a-1}{a}a^n\left(1+\sum_{l\geq 1}(-1)^lZ^l\right). \tag{3.8}$$

We recognize in (3.8) the main term from (3.2). To obtain the other terms, we expand Z^l using the multinomial formula.

$$\begin{aligned}
Z^l &= \left(\sum_{k\geq 1} f_k \left(\frac{b}{a}\right)^{kn}\right)^l \\
&= \sum_{\substack{i_1\geq 0,\ldots,i_l\geq 0 \\ i_1+\cdots+i_l=l}} \sum_{1\leq k_1<k_2<\cdots<k_l} \binom{l}{i_1,\ldots,i_l} f_{k_1}^{i_1}\cdots f_{k_l}^{i_l}\left(\frac{b}{a}\right)^{(i_1k_1+\cdots+i_lk_l)n} \\
&= \sum_{k\geq 1} f_{k,l}\left(\frac{b}{a}\right)^{kn},
\end{aligned}$$

where

$$f_{k,l} = \sum_{\substack{i_1\geq 0,\ldots,i_l\geq 0 \\ i_1+\cdots+i_l=l \\ 1\leq k_1<k_2<\cdots<k_l \\ k_1i_1+\cdots+k_li_l=k}} \binom{l}{i_1,\ldots,i_l} f_{k_1}^{i_1}\cdots f_{k_l}^{i_l}. \tag{3.9}$$

Hence, by (3.8)-(3.9), we obtain (for sufficiently large n)

$$\begin{aligned}
h_n &= c\frac{a-1}{a}a^n + c\frac{a-1}{a}\sum_{k\geq 1}\sum_{l\geq 1}(-1)^l f_{k,l}\left(\frac{b^k}{a^{k-1}}\right)^n, \\
&= c\frac{a-1}{a}a^n + c\frac{a-1}{a}\sum_{k=1}^{k_0} e_k\left(\frac{b^k}{a^{k-1}}\right)^n + c\frac{a-1}{a}\sum_{k>k_0} e_k\left(\frac{b^k}{a^{k-1}}\right)^n,
\end{aligned} \tag{3.10}$$

with

$$e_k = \sum_{l\geq 1}(-1)^l f_{k,l}, \quad k\geq 1. \tag{3.11}$$

To complete the proof of (3.2) we need to make two observations.

First note, that for each fixed k, the apparently infinite sum over l in (3.9) and (3.11) in fact stops at $l=k$. To see this, notice that if $i_1k_1+\cdots+i_lk_l=k$, with $1\leq k_1<\cdots<k_l$, then

$$l = i_1+\cdots+i_l \leq i_1k_1+\cdots+i_lk_l = k.$$

Second, we must estimate the third summand on the right hand side of (3.10).

First we fix $k>k_0$ and, using (3.9) and (3.11), estimate $|e_k|$, the coefficient of $a^n\left(\frac{b}{a}\right)^{kn}$.

The index of summation in (3.9) counts partitions of k with i_1 parts equal to k_1, i_2 parts equal to k_2 and so on up to i_l parts equal to k_l. The multinomial coefficient counts the number of ways of permuting these parts within the sum. Hence, this quantity is the same as the number of ways of writing $k_1+\cdots+k_l=k$, with positive integers $k_1,\ldots,k_l$ and since this number equals $\binom{k}{l-1}$ the sum of all these binomial coefficients over all possible l is bounded above by 2^{k-1}.

By (3.9) and (3.5),

$$f_{k_1}^{i_1}f_{k_2}^{i_2}\cdots f_{k_l}^{i_l} < \left(\left|\frac{d}{c}\right|\right)^{k_1i_1+k_2i_2+\cdots+k_li_l} = \left(\left|\frac{d}{c}\right|\right)^k.$$

Combining these two estimates it follows, that for large enough n, say for all $n > n_0$, that an estimate of the third summand on the right-hand side of (3.10) is

$$\begin{aligned} c\frac{a-1}{a}\sum_{k>k_0} e_k \left(\frac{b^k}{a^{k-1}}\right)^n &< c\frac{a-1}{a}a^n \sum_{k\geq k_0+1} 2^{k-1}\left(\left|\frac{d}{c}\right|\right)^k \left(\frac{|b|}{a}\right)^{kn} \\ &< c\frac{a-1}{a}\left(\left|\frac{2d}{c}\right|\right)^{k_0+1}\left(\frac{|b|^{k_0+1}}{a^{k_0}}\right)^n \sum_{k\geq 0}\left(\left|\frac{2d}{c}\right|\left(\frac{|b|}{a}\right)^n\right)^k \qquad (3.12) \\ &< O(1), \end{aligned}$$

where n_0 is large enough so that $\left|\frac{2d}{c}\right|\left(\frac{|b|}{a}\right)^{n_0} < 1$. Such a choice is possible, since by (1.8) $\frac{|b|}{a} < 1$. The $O(1)$ result now follows from (3.1).

Note the subtlety throughout the proof, that certain constants must be independent of both k and n.

Equation (3.2) now follows from (3.10) and (3.12). □

Comment: Note that the $O(1)$ in (3.2) which come from (3.12) is in fact $o(1)$ except if $a^{k_0-1} = |b|^{k_0}$, which can happen for example when $\{G_n\}_{n\geq 0}$ are the Fibonacci numbers.

4. Consequences of the Main Theorem

In this section we produce explicit computations and formulate consequences of the main theorem for almost-recursiveness. The following well-known fact about recursive sequences facilitates statements of corollaries to the main theorem.

Proposition 4.1. *Suppose $r_i, 1 \leq i \leq m, m \in \mathbb{N}$, are distinct reals and $g_i, 1 \leq i \leq m$ are arbitrary reals (not necessarily distinct from each other). Then the sequence,*

$$J_n = \sum_{i=1}^{m} g_i r_i^n,$$

satisfies the recursion $J_n = \sum_{i=1}^{m} m_i J_{n-i}$, with the m_i defined by

$$X^m - \sum_{i=1}^{m} m_i x^{m-m_i} = \prod_{i=1}^{m}(x - r_i).$$

Corollary 4.2. *$\{G_t\}_{t\geq 0}$ is almost-recursive with respect to any recursion satisfying the recursive sequence $c\frac{a-1}{a}a^n + c\frac{a-1}{a}\sum_{k=1}^{k_0} e_k\left(\frac{b^k}{a^{k-1}}\right)^n$.*

Proof. Using the notation of Proposition 4.1 and Theorem 3.1, let $m = k_0 + 1$, $r_1 = a$, $g_1 = c\frac{a-1}{a}$, and for $1 \leq i \leq k_0$, let $r_{i+1} = \frac{b^i}{a^{i-1}}$, and $g_{i+1} = c\frac{a-1}{a}e_i$. The result now follows by (1.2). □

The following identity is useful in computations. By (3.9),

$$f_{k,1} = f_k, k = 1, 2, \ldots. \qquad (4.1)$$

Corollary 4.3. *If (1.8) and (2.1) holds then $\{G_t\}_{t\geq 0}$ is almost-recursive with respect to itself.*

Proof. By (3.1) and the assumptions, we have $k_0 = 1$. By (3.2), (3.4), (3.9), (3.11), and (4.1), we have

$$h_n = c\frac{a-1}{a}a^n + d\frac{(a-1)^2}{a^2-b}b^n + O(1). \tag{4.2}$$

Proposition 4.1 asserts that both $G_n = ca^n + db^n$ and $c\frac{a-1}{a}a^n + d\frac{(a-1)^2}{a^2-b}b^n$, satisfy a second order recursion whose coefficients are given by the non-monic terms in the polynomial, $(x-a)(x-b) = x^2-(a+b)x+ab$. Therefore, $\{G_t\}_{t\geq 0}$ is recursive with respect to itself. □

Corollary 4.4. *The Fibonacci numbers are almost-recursive with respect to themselves* [1].

Proof. In the main theorem, let $c = \frac{1}{\sqrt{5}}$, $a = \frac{\sqrt{5}+1}{2}$, $b = \frac{-\sqrt{5}+1}{2}$, and $d = -\frac{1}{\sqrt{5}}$. Then $\{G_n\}_{n\geq 0} = \{F_n\}_{n\geq 0}$. By (3.1), $k_0 = 0$. The main theorem implies that the Fibonacci numbers are almost-recursive with respect to any recursion satisfying the sequence $\{c\frac{a-1}{a}a^n\}_{n\geq 0}$. It immediately follows that the Fibonacci numbers are almost-recursive with respect to themselves, since $\{F_n\}_{n\geq 0}$ and $\{c\frac{a-1}{a}a^n\}_{n\geq 0}$ both satisfy the Fibonacci recursion. □

Equation (4.2) describes $\{h_n\}_{n\geq 1}$ when $k_0 = 1$, in Theorem 3.1. The next proposition deals with the case $k_0 = 2$. It is an interesting exercise to prove similar Propositions for the cases $k = 3, 4, \ldots$. However, for this paper, we suffice with Corollaries 4.3 and 4.5.

Corollary 4.5. *If in* (1.8), $k_0 = 2$, *then*

$$h_n = c\frac{a-1}{a}a^n + d\frac{(a-1)^2}{a^2-b}b^n + \frac{(a-1)^2ad^2}{c}\left(\frac{a-1}{(a^2-b)^2} - \frac{1}{a^3-b^2}\right)\left(\frac{b^2}{a}\right)^n + O(1). \tag{4.3}$$

Proof. By (3.2), (3.4), (3.9), (3.11), and (4.1). □

Example 4.6. *In the Main Theorem let $c = 1$, $a = 3$, $b = 2$, and $d = 1$. Then, by (1.8), $G_n = 3^n + 2^n$. By (3.1), since $b^2 > a$ but $b^3 < a^2$, $k_0 = 2$. Using Proposition 4.5 we compute*

$$h_n\left(\{3^m + 2^m\}_{m\geq 0}\right) = \frac{2}{3}\cdot 3^n + \frac{4}{7}\cdot 2^n - \frac{36}{1127}\left(\frac{4}{3}\right)^n + O(1).$$

$\{G_n\}_{n\geq 0}$ is almost-recursive with respect to any recursion satisfying the sequence determined by the first three summands on the right hand side, whose first few terms are

$$\{3.1, 8.2, 22.5, 63.0, 180.2, 522.4, \ldots\}.$$

Note that this sequence is approximated well by $\{H_n\}_{n\geq 1} = \{3, 8, 23, 63, 180, 522, \ldots\}$. The coefficients for a recursion for $\{G_n\}_{n\geq 0}$ may be obtained by expanding the polynomial

$$(X-3)(X-2)\left(X - \frac{4}{3}\right).$$

We can then compute the sequence $\{3t_n\}_{n\geq 4} = \{-16, 25, -12, 3, -13, 12, 3, -13, 12, 0, 0, \ldots\}$. As indicated earlier in the paper, although this sequence is bounded, it does not seem to exhibit periodicity.

5. Acknowledgement

The author acknowledges useful conversations with Florian Luca which led to a more general statement of the Main Theorem. The author also acknowledges an anonymous referee who significantly improved the appearance of the manuscript and who also provided a computationally explicit version of the main theorem.

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MSC2010: 11B37, 11B39

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