ADDITIVE PROPERTIES OF THE FIBONACCI SEQUENCE

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Abstract. In this note, we shall study some additive properties of the Fibonacci sequence F. In particular, we prove a lower and an upper bound for the quantity $\sup_{A\subseteq\mathbb{N}}\{\overline{\mathrm{d}}(A) : (A-A)\cap F=\emptyset\}$.

1. Introduction

Let $\mathbb{N}=A_1\cup\cdots\cup A_k$ be any $k-$partition of natural numbers. Denote by $E=E(A_1,\dots,A_k)$ the set of all positive even integers, which cannot be written as the sum of two different monochromatic numbers. Erdős, Sarközy, and Sós [2] proved that for every $2-$coloring of $\mathbb{N}$ we have

$$E(n)=|E\cap[n]|\le\frac{\log n}{\log((1+\sqrt{5})/2)}\tag{1.1}$$

for every $n\in\mathbb{N}$. They also showed that there is a $2-$partition of $\mathbb{N}=A\cup B$ such that $2^n\in E$ for all $n\in\mathbb{N}$. Thus,

$$E(A,B)(n)\ge\lfloor\log_2 n\rfloor.$$

Furthermore, for any $k-$coloring they obtained the following estimate

$$E(n)\le 3n^{1-2^{-k-1}},\tag{1.2}$$

so that considering the finite version of the problem $[n]=A_1\cup\cdots\cup A_k$ the exceptional set has size $o(n)$ for every $k=o(\log\log n)$. This solves a problem posed by Roth (see problem E9 in [3]).

In the first part of this note we prove that the inequality (1.1) is the best possible. More precisely, we will provide a simple construction of a $2-$coloring of $\mathbb{N}$ such that no number $G_n=2F_n$ (where $(F_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence) has a monochromatic representation. We will also improve the inequality (1.2) for $\log\log n\ll k=o((\log n)^{2/3+o(1)})$. In the later part of the paper we consider the intersection property of the Fibonacci sequence. We show that every set $A\subseteq\mathbb{N}$ such that $(A-A)\cap\{F_1,F_2,\dots\}=\emptyset$ has lower density smaller than 7/36. On the other hand, we prove that there exists such a set with density 19/110. With some effort one can improve both bounds using the same argument, however we are not able to obtain the best possible estimate for the density of sets with such property.

We keep the following notation. Let $\mathbb{N}$ stand for the set of all positive integers. For a set $A\subseteq\mathbb{N}$ its counting function is denoted by $A(n):=|A\cap[n]|$, where $[n]=\{1,\dots,n\}$. For a real number x we define $||x||$ as the distance from the nearest integer number. Furthermore $\underline{\mathrm{d}}(A)$ and $\overline{\mathrm{d}}(A)$ stand for the lower and the upper density of A, respectively. By $A+B$ we mean the set of all numbers represented in the form $a+b$, $a\in A$, $b\in B$ and $A\hat{+}B$ is the set of numbers

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written as a sum of distinct elements. Finally, by F we denote the set of all Fibonacci numbers.

2. Monochromatic Sums

Our first result shows that one cannot improve (1.1). Our construction is quite straightforward and makes use of Fibonacci numbers.

Theorem 2.1. *There exists a* $2-$*partition* $\mathbb{N} = A \cup B$ *such that*

$$G_n = 2F_n \in E(A, B), \tag{2.1}$$

where $(F_n)_{n\in\mathbb{N}}$ *is the Fibonacci sequence.*

Proof. We construct recursively two sequences of sets A_n and B_n such that $A_n \cup B_n = [G_n]$ and

$$A_n \subseteq A_{n+1},\ B_n \subseteq B_{n+1}$$

for all $n \in \mathbb{N}$. We start with $A_1 = A_2 = \{1\}$, $B_1 = B_2 = \{2\}$, and $A_3 = \{1, 4\}$, $B_3 = \{2, 3\}$. Clearly, no G_k has a monochromatic representation. Now assume that A_n and B_n such that $A_n \cup B_n = [G_n]$ and

$$G_k \notin (A_n \hat{+} A_n) \cup (B_n \hat{+} B_n),\ k \in \mathbb{N}$$

have already been defined. Let

$$\begin{aligned} A_{n+1} &= A_n \cup \{G_{n+1} - x : x \in B_n \cap [G_{n-1} - 1]\} \cup S_A, \\ B_{n+1} &= B_n \cup \{G_{n+1} - x : x \in A_n \cap [G_{n-1} - 1]\} \cup S_B, \end{aligned} \tag{2.2}$$

where

$$S_A = \begin{cases} \{G_{n+1}\}, & \text{if } G_n \in B_n, \\ \emptyset, & \text{if } G_n \notin B_n. \end{cases}$$

$$S_B = \begin{cases} \{G_{n+1}\}, & \text{if } G_n \in A_n, \\ \emptyset, & \text{if } G_n \notin A_n. \end{cases}$$

We have to check that no number G_k possesses a monochromatic representation. Since $G_{k+2} > 2G_k$ for all $k \in \mathbb{N}$ and $A_n \subseteq A_{n+1}$, $B_n \subseteq B_{n+1}$ it is enough to show that $G_{n+1}, G_{n+2} \notin (A_{n+1} \hat{+} A_{n+1}) \cup (B_{n+1} \hat{+} B_{n+1})$. Suppose that there are integers $x, y \in A_{n+1}, x > y$, (the case $x, y \in B_{n+1}$ is identical) such that $x + y = G_{n+1}$. However, it follows from the definition of A_{n+1} that $x, y < G_n$, hence, $x, y \in A_n$, contradicting the inductive assumption. Now suppose that for some $x, y \in A_{n+1}, x > y$, we have $x + y = G_{n+2}$. Clearly $G_n < y < x < G_{n+1}$, and by (2.2), $G_{n+1} - x, G_{n+1} - y \in B_n$. Therefore,

$$(G_{n+1} - x) + (G_{n+1} - y) = 2G_{n+1} - G_{n+2} = G_{n+1} - G_n = G_{n-1},$$

which is again impossible.

Finally, letting

$$A = \bigcup_{n\in\mathbb{N}} A_n, \quad B = \bigcup_{n\in\mathbb{N}} B_n$$

we obtain a $2-$partition with required property. $\square$

Next, the results show that in the finite version of the problem we have $|E(A_1, \ldots, A_k)| = o(n)$ for all $k = o((\log n)^{3/4+o(1)})$. Denote by $\nu_3(n)$ the maximal size of a subset of $[n]$ not containing any nontrivial three term arithmetic progression.

Theorem 2.2. *Let* $[n] = A_1 \cup \cdots \cup A_k$ *be a* $k-$*partition. Then*

$$E(A_1, \ldots, A_k)(2n) \leq k\nu_3(n). \tag{2.3}$$

Proof. Let $E/2 = \{e/2 : e \in E\}$ and put $E_i = E/2 \cap A_i, i = 1, 2, \ldots, k$. We show that every set E_i does not contain any nontrivial arithmetic progression of length three. Indeed, suppose that $a,\ a + d,\ a + 2d \in E_i$ for some $a, d \in \mathbb{N}$ and $1 \leq i \leq k$. Then $2a + 2d \in E$ has a monochromatic representation

$$2a + 2d = a + (a + 2d) \in A_i \hat{+} A_i,$$

which is a contradiction. Hence, for every $1 \leq i \leq k$

$$|E_i| \leq \nu_3(n)$$

and the result follows. □

The best known upper bound on ν_3 is due to Sanders [4], who showed that

$$\nu_3(n) \ll n/(\log n)^{3/4+o(1)}.$$

Thus by Theorem 2.2, $E(A_1, \ldots, A_k)(n) \ll kn/(\log n)^{2/3+o(1)}$, so that $E(A_1, \ldots, A_k)(n) = o(n)$, for $k = o((\log n)^{3/4+o(1)})$.

Finally, we observe that (1.2) can be improved for $k = 3$. It follows from the proof of Theorem 4 in [1] that there is a constant $C > 0$ such that for every set $E \subseteq [n]$ of size $Cn^{1/2}$ of even numbers there exist distinct positive integers a_1, a_2, a_3, a_4 with all sums $a_i + a_j, 1 \leq i < j \leq 4$, in E. Therefore, the upper bound $3n^{15/16}$ in (1.2) can be replaced by $Cn^{1/2}$.

3. Intersection Properties of the Fibonacci Sequence

In this section we shall study intersection properties of the Fibonacci sequence. More precisely, we prove a lower and an upper bound for the quantity

$$\sup_{A \subseteq \mathbb{N}} \{\overline{\mathrm{d}}(A) : (A - A) \cap F = \emptyset\}.$$

Notice that, if $(A - A) \cap F = \emptyset$, then sets

$$A, A + 2, A + 3, A + 5$$

are pairwise disjoint. This shows that $\overline{\mathrm{d}}(A) \leq 1/4$. One can easily improve this bound, observing that if $a \in A$, then at most one among elements $a + 1, \ldots, a + 9$ belongs to A, so that $\overline{\mathrm{d}}(A) \leq 1/5$. Our next theorem provides further refinement of the above inequality.

Theorem 3.1. *Suppose that* $A \subseteq \mathbb{N}$ *and* $(A - A) \cap F = \emptyset$. *Then*

$$\overline{\mathrm{d}}(A) \leq \frac{7}{36} = 0.19(4).$$

Proof. Since the proof consists of many similar cases, which can be treated in the same way, we will not present all details. Let A be a set such that $(A - A) \cap F = \emptyset$ and let a be an arbitrary element of A. Then, clearly none of the numbers $a + 1$, $a + 2$, $a + 3$, $a + 5$ belong to A. If $a + 4 \notin A$, then the "local" density of A in the interval $[a, a + 5]$ is $1/6$, so we assume that $a+4 \in A$. Hence, $\{a+5, a+6, a+7\} \cap A = \emptyset$ and in view of $5, 8 \in F$ also $a+8, a+9 \notin A$. We can assume that $a + 10 \in A$, otherwise A contains at most 2 integers from the interval $[a, a + 10]$. If $a + 14 \notin A$ then again the "local" density of A on the interval $[a, a + 15]$ is $3/16$. Thus, we assume that $a+14 \in A$. Then, clearly $\{a+11, a+12, a+13, a+15, a+16, a+17, a+18\} \cap A = \emptyset$. Using a similar argument (and the fact that $34 \in F$) one can show that

$$A \cap [a, a+35] = \{a, a+4, a+10, a+14, a+20, a+24, a+30\},$$

otherwise there is a positive integer $i \leq 35$ such that

$$|A \cap [a, a+i]| \leq \frac{7}{36}(i+1).$$

Therefore, for every positive integer n one can find a sequence of elements $a_1, \ldots, a_k \in A \cap [n]$ and a sequence of integers $i_1, \ldots, i_k \in [35]$ such that $a_1 = \min A$, a_{j+1} is the smallest element of A greater than $a_j + i_j$, $[a_k + i_k + 1, n] \cap A = \emptyset$ and

$$|A \cap [a_j, a_j + i_j]| \leq \frac{7}{36}(i_j + 1).$$

Thus,

$$|A \cap [n]| = \sum_{j=1}^{k} |A \cap [a_j, a_j + i_j]| \leq \sum_{j=1}^{k} \frac{7}{36}(i_j + 1) \leq \frac{7}{36}(n + 35),$$

so that

$$\overline{\mathrm{d}}(A) \leq \frac{7}{36}.$$

$\square$

Our last theorem shows that Theorem 3.1 is not far from the best possible.

Theorem 3.2. *There exists a set $A_0 \subseteq \mathbb{N}$ such that $(A_0 - A_0) \cap F = \emptyset$ and*

$$\mathrm{d}(A_0) = \frac{19}{110} = 0.1(72).$$

To prove Theorem 3.2, we will need the following lemmas.

Lemma 3.3 (Ruzsa, Tuza, Voigt [5])**.** *Let $(d_i)_{i \in \mathbb{N}}$ be a sequence such that $d_{i+1} \geq \alpha d_i$ for some $\alpha > 1$ and all i. Then there exists a real number x such that for every $i \geq 1$*

$$\|x d_i\| \geq \delta := \frac{1}{2} - \frac{1}{2\alpha - 2}.$$

Since we will not directly apply Lemma 3.3, we briefly sketch the idea of its proof. Using lacunarity of $(d_i)_{i \in \mathbb{N}}$ one can show that there exists a sequence of integers $(z_i)_{i \in \mathbb{N}}$ such that

$$I_{i+1} = \left[\frac{z_{i+1} + \delta}{d_{i+1}}, \frac{z_{i+1} + 1 - \delta}{d_{i+1}}\right] \subseteq I_i = \left[\frac{z_i + \delta}{d_i}, \frac{z_i + 1 - \delta}{d_i}\right] \tag{3.1}$$

for all $i \in \mathbb{N}$. Then, for every $y \in I_i$ we have $\|y d_i\| \geq \delta$ $(i = 1, 2, \ldots)$. Thus x is the unique element in $\bigcap_{j=0}^{\infty} I_j$. Furthermore, it follows from the proof that the choice of z_{i+1} is arbitrary, provided that (3.1) is satisfied.

For the Fibonacci sequence $(F_n)_{n \in \mathbb{N}}$ one can take $\alpha = 3/2$, which is too small to prove Theorem 3.2. Therefore, we consider the subsequence $(F_{3n+3})_{n \in \mathbb{N}}$ of even Fibonacci numbers.

Lemma 3.4. *For every positive integer n we have*

$$F_{3n+6} \geq 4\frac{4}{17} F_{3n+3}. \tag{3.2}$$

Proof. If $n = 1$ we have

$$F_9 = 34 \geq 4\frac{4}{17} \cdot F_6 = 33\frac{15}{17}.$$

For $n \geq 2$. (3.2) is equivalent to

$$\begin{aligned} 34F_{3n+2} &\geq 21F_{3n+3} \\ 13F_{3n+2} &\geq 21F_{3n+1} \\ 13F_{3n} &\geq 8F_{3n+1} \\ 5F_{3n} &\geq 8F_{3n-1} \\ 5F_{3n-2} &\geq 3F_{3n-1} \\ 2F_{3n-2} &\geq 3F_{3n-3} \\ 2F_{3n-4} &\geq F_{3n-3} \\ F_{3n-4} &\geq F_{3n-5}. \end{aligned}$$

The last inequality is clearly satisfied. □

Proof of Theorem 3.2. By Lemma 3.3 and 3.4 there exists a real number x such that

$$\|xF_{3n+3}\| \geq \delta = 19/55 \tag{3.3}$$

for every $n \geq 1$. We have to check that $F_3 = 2$ also satisfies the above inequality. It will be done if we show that there is an integer z_0 such that

$$I_n = \left[\frac{z_n + \delta}{F_{3n+3}}, \frac{z_n + 1 - \delta}{F_{3n+3}}\right] \subseteq I_0 = \left[\frac{z_0 + \delta}{2}, \frac{z_0 + 1 - \delta}{2}\right]$$

for some $n \geq 1$. As mentioned before, we can choose any z_{i+1} provided that (3.1) is fulfilled. Taking $z_1 = 1$, $z_2 = 6$ we have

$$I_2 = \left[\frac{349}{1870}, \frac{366}{1870}\right] \subseteq I_1 = \left[\frac{74}{440}, \frac{91}{440}\right].$$

For $z_0 = 1$, obviously

$$I_2 = \left[\frac{349}{1870}, \frac{366}{1870}\right] \subseteq I_0 = \left[\frac{19}{110}, \frac{91}{110}\right],$$

hence $x \in I_0$ and (3.3) holds for every $n \geq 0$.

Finally, we define A_0 by

$$A_0 = \{k \in 2\mathbb{N} \colon \|xk\| < \delta/2\}.$$

Then, clearly $A_0 - A_0$ shares no element with F.

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