A CHARACTERIZATION OF CONVERGING DUCCI SEQUENCES OVER $\mathbb{Z}_2$

CHRISTIAN AVART

Abstract. It is well-known that any Ducci sequence generated by a vector of length a power of 2 will eventually reach the null vector. As an easy consequence, all vectors obtained by concatenating several copies of a vector of length 2 eventually reach the null vector. We prove a converse to this statement for Ducci sequences over the field $\mathbb{Z}_2$. Namely that over $\mathbb{Z}_2$, the only vectors converging to the null vector are the vectors obtained by concatenation of several copies of a vector of length a power of 2.

1. Introduction to Ducci Sequences

Let $k \in \mathbb{N}$ and let $\vec{x} = (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{N}^k$. We define a map $T : \mathbb{N}^k \to \mathbb{N}^k$ by

$$T(\vec{x}) = T(a_0, a_1, \ldots, a_{k-1}) = (|a_0 - a_1|, |a_1 - a_2|, \ldots, |a_{k-1} - a_0|).$$

The sequence $(T^n(\vec{x}))_{n\in\mathbb{N}}$ generated by the iterations of T is called a *Ducci sequence*. Ducci sequences have been extensively studied and often rediscovered. We now mention a well-known result motivating the work presented in this paper.

Let $\vec{x} = (a_0, a_1, \ldots a_{k-1}) \in \mathbb{N}^k$. If there exists $a \in \mathbb{N}$ such that for every $0 \leq i \leq k$, $a_i \in \{0, a\}$, we will say that $\vec{x}$ is a *simple* vector. A well-known result states that for every $\vec{x} \in \mathbb{N}^k$, there exists $n \in \mathbb{N}$ such that $T^n(\vec{x})$ is simple. We derive an important consequence of this fact.

Let $\vec{x} = (a_0, a_1, \ldots, a_{k-1})$ be a simple vector whose coordinates are in $\{0, a\}$. First, rewrite $\vec{x}$ as $(a \cdot \epsilon_0, a \cdot \epsilon_1, \ldots, a \cdot \epsilon_{k-1})$, where $\epsilon_i = 1$ if $a_i = a$ and 0, otherwise. Notice that for every $k \in \mathbb{N}$, $T^k(a \cdot \epsilon_0, a \cdot \epsilon_1, \ldots, a \cdot \epsilon_{k-1}) = a \cdot T^k(\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1})$. Since every Ducci sequence eventually reaches a simple vector this implies that in order to study the asymptotical behavior of Ducci sequences, it is sufficient to study vectors with coordinates in $\{0, 1\}$. Note also that when $a_i \in \{0, 1\}$, the operation $|a_i - a_{i+1}|$ is equivalent to $a_i + a_{i+1} \pmod 2$. This justifies the importance of the map $T : Z_2^k \to \mathbb{Z}_2^k$ defined by

$$T(a_0, a_1, \ldots, a_{k-1}) = (a_0 + a_1, a_1 + a_2, \ldots, a_{k-1} + a_0).$$

We will call the sequences generated by iteration of this map *Ducci sequences over* $\mathbb{Z}_2$.

2. Notation and Result

In order to simplify the notation, the indices of the coordinates of any vector $\vec{x} \in \mathbb{N}^k$ will be written modulo k so that, for example, $a_k = a_0$ and $a_{k+1} = a_1$.

If for some integer m $T^m(\vec{x}) = \vec{x}$, we say that $\vec{x}$ is cyclic. If m is the smallest such integer, we say that $\vec{x}$ is m-cyclic. If for some integer m we have $T^m(\vec{x}) = \vec{y}$ and $\vec{y}$ is cyclic, we say that $\vec{y}$ *belongs to the cycle generated by* $\vec{x}$. Note that $\vec{x}$ does not necessarily belong to the cycle generated by itself. If for some integer m $T^m(\vec{x}) = \vec{0}$, we say that $\vec{x}$ is *nilpotent*.

Given $k, l \in \mathbb{N}$ and two vectors $\vec{x} = (x_0, x_1, \dots, x_k)$ and $\vec{y} = (y_0, y_1, \dots, y_l)$ we denote by $\vec{x} \vee \vec{y}$ their concatenation $(x_0, x_1, \dots, x_k, y_1, y_2, \dots, y_l)$. We also write $\vec{x} \vee \vec{x} \dots \vee \vec{x} = \vee^{(m)}\vec{x}$, the concatenation of $\vec{x}$ with itself m times and $\vee^{(1)}\vec{x} = \vec{x}$.

It is easy to see that for any $\vec{x} \in \mathbb{Z}_2^k$ and for any positive integers n, m the following relation holds:

$$T^n(\vee^{(m)}\vec{x}) = \vee^{(m)}T^n(\vec{x}). \tag{2.1}$$

In particular, $T^n(\vec{x}) = \vec{0}$ implies $T^n(\vee^{(m)}\vec{x}) = \vec{0}$ for every m.

It is well-known that if $k = 2^l$ for some l, then any $\vec{x} \in \mathbb{Z}_2^k$ is nilpotent. Together with (2.1), this implies that for any $m \in \mathbb{N}$ and any $\vec{x} \in \mathbb{Z}_2^{2^l}$, the vector $\vee^{(m)}\vec{x}$ is nilpotent. The goal of this paper is to prove that any nilpotent vector in $\mathbb{Z}_2$ is obtained this way.

Proposition 2.1. *Let $\vec{x} \in \mathbb{Z}_2^k$ be a nonzero vector. If $\vec{x}$ is nilpotent, there exist $l, m \in \mathbb{N}$ and $\vec{y} \in \mathbb{Z}_2^{2^l}$ such that $\vec{x} = \vee^{(m)}\vec{y}$.*

3. Proof of Proposition 2.1

Proof. For any vector $\vec{x} = (a_0, a_1, \dots, a_{k-1}) \in \mathbb{Z}_2^k$ we define $r(\vec{x}) = (a_1, a_2, \dots, a_{k-1}, a_0)$. Observe that $T(\vec{x}) = \vec{x} + r(\vec{x})$. We obtain by induction

$$T^n(\vec{x}) = \sum_{i=0}^{n} \binom{n}{i} r^i(\vec{x}).$$

Now suppose $T^n(\vec{x}) = \vec{0}$. Then for any l such that $2^l \geq n$ we have $T^{2^l}(\vec{x}) = \vec{0}$. Combining with the previous equality and using the fact that $\binom{2^l}{i} \equiv 0 \pmod 2$ for $1 \leq i \leq 2^l - 1$ we obtain

$$T^{2^l}(\vec{x}) = \sum_{i=0}^{2^l} \binom{2^l}{i} r^i(\vec{x}) = \vec{x} + 0 + 0 + \cdots + 0 + r^{2^l}(\vec{x}) = \vec{0}.$$

In other words $\vec{x} = r^{2^l}(\vec{x})$ and for every $i \in [k]$, $a_i = a_{i+2^l}$.

Consider the Euclidean division of 2^l by k, $2^l = mk + r$, where $0 \leq r < k$ is the remainder. Then $a_i = a_{i+2^l} = a_{i+mk+r} = a_{i+r}$, i.e., the coordinates of $\vec{x}$ are r-periodic. Let $d = (k, r)$, the greatest common divisor of k and r. By Bezout's Theorem there exists $a, b \in \mathbb{Z}$ such that $ak + br = d$. Since the coordinates of $\vec{x}$ are k-periodic and r-periodic we have $a_i = a_{i+ak} = a_{i+ak+br} = a_{i+d}$ so that the coordinates of $\vec{x}$ are d-periodic. Since $d|k$ and $d|r$ we also have $d|mk + r = 2^l$ so that $d = 2^{l'}$ for some $l' \leq l$. Since $d|k$ and $\vec{x}$ is d-periodic, $\vec{x} = \vee^{(m)}\vec{y}$ for $\vec{y} = (a_0, a_1, \dots, a_{2^{l'}-1})$, concluding the proof. $\square$

4. Conclusion

By combining Proposition 2.1 with the remark mentioned in the introduction, we obtain the following theorem.

Theorem 4.1. *A vector $\vec{x} \in \mathbb{Z}_2^k$ is nilpotent if and only if it can be written as the concatenation of several copies of a vector of length a power of 2.*

Note that Proposition 2.1 also provides a necessary condition for a vector in $\mathbb{N}^k$ to be nilpotent. Let $\vec{x} = (a_0, a_1, \dots, a_{k-1})$ be a vector in $\mathbb{N}^k$ and consider the vector $\vec{y} = (b_0, b_1, \dots, b_{k-1})$ where $b_i = a_i \pmod 2$. If $\vec{x}$ is nilpotent, so is $\vec{y}$ over $\mathbb{Z}_2$. By Proposition 2.1 this implies that $\vec{y}$ is the concatenation of vectors of length a power of 2, which in return gives us a condition on the parity of the a_i's.

References

[1] A. Ehrlich, *Periods in Ducci's n-number game of differences*, The Fibonacci Quarterly, **26.2** (1990), 302–305.
[2] H. Glaser and G. Schoffl, *Ducci sequences and Pascal's triangle*, The Fibonacci Quarterly, **33.4** (1995), 313–324.
[3] A. Ludington Furno, *Cycles of differences of integers*, J. Number Theory, **13** (1981), 255–261.
[4] A. Ludington-Young. *Length of the n-number game*, The Fibonacci Quarterly, **28.3** (1990), 259–265.

MSC2010: 11B02

Department of Mathematics and Statistics, Georgia State University, Atlanta, Georgia 30303
E-mail address: matcxa@langate.gsu.edu