ON THE DISTRIBUTION OF THE EULER FUNCTION WITH FIBONACCI NUMBERS

JEAN-MARC DESHOUILLERS AND FLORIAN LUCA

Abstract. Here, we show that the distribution function of $\phi(F_n)/F_n$ is strictly increasing in the interval $[0,1]$. We also show that the summatory function of $\phi(F_n)/F_n$ is dense modulo 1.

1. Introduction

Let $\phi(n)$ be the Euler function of the positive integer n. It is well-known that $\phi(n)/n$ has a distribution function; that is, for each real number $u \in [0,1]$, the density

$$D_\phi(u) = \lim_{x\to\infty} \frac{1}{x}\#\{n \le x : \phi(n)/n \le u\}$$

exists. Moreover, $D_\phi(0) = 0$, $D_\phi(1) = 1$, and the function $D_\phi(u)$ is strictly increasing and continuous. This was proved by Schoenberg [9]. This result is often said to mark the dawn of probabilistic number theory. In this paper, we look at the distribution function for $\phi(F_n)/F_n$, where $(F_m)_{m\ge 1}$ is the Fibonacci sequence given by $F_1 = 1$, $F_2 = 1$ and $F_{m+2} = F_{m+1} + F_m$ for all $m \ge 1$. There are quite a few papers in the literature treating problems concerning the distribution function for the Euler function with linear recurrence arguments. For example, by modifying appropriately the proof of Theorem 3 in [6], it follows that for each $u \in [0,1]$, the density

$$D_F(u) := \lim_{x\to\infty} \frac{1}{x}\#\{n \le x : \phi(F_n)/F_n \le u\}$$

exists. Moreover, $D_F(0) = 0$, $D_F(1) = 1$ and the function $D_F(u)$ is nondecreasing and continuous. It has not been proved before that $D_F(u)$ is strictly increasing. A weaker result, namely that $\{\phi(F_n)/F_n\}_{n\ge 1}$ is dense in $[0,1]$ appears in Section 4 of [7]. Here, we prove that this is indeed the case.

Theorem 1.1. *The function $D_F(u)$ is strictly increasing in the interval* $[0,1]$.

A study of multiplicative functions, the distributions of which are strictly increasing, is the topic of [2].

In the previous paper [3], we proved that various means involving the Euler function in the interval $[1,m]$ are dense modulo 1. More specifically, we showed that the sequences of general terms

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$$s_n := \sqrt{\sum_{m\le n}\phi(m)}, \qquad a_n := \frac{1}{n}\sum_{m\le n}\phi(m),$$
$$g_n := \left(\prod_{m\le n}\phi(m)\right)^{1/n}, \qquad h_n := \frac{n}{\sum_{m\le n}\frac{1}{\phi(m)}},$$

respectively, are all dense modulo 1. The first three of them have a linear growth, while the last one has sub–linear growth which is why its study is easier. In the subsequent paper [4], H. Iwaniec together with the first author showed that the sequence $(a_n)_{n\ge1}$ is in fact uniformly distributed modulo 1. In the paper [5], it was shown that the sequence of general term

$$c_n := \sum_{m\le n}\frac{\phi(m^2+1)}{m^2+1} \tag{1.1}$$

is also dense modulo 1. Here, we treat a sequence similar to $(c_n)_{n\ge1}$, but where the polynomial input m^2+1 is replaced by the mth Fibonacci number F_m. We have the following result.

Theorem 1.2. *The sequence of general term*

$$f_n := \sum_{m\le n}\frac{\phi(F_m)}{F_m}$$

is dense modulo 1.

We point out that it follows as a special case of a result from [8], that

$$\frac{f_n}{n} = \Gamma_f + O\left(\frac{(\log\log n)^2}{\log n}\right),$$

where

$$\Gamma_f := \sum_{d\ge1}\frac{\mu(d)}{dz(d)},$$

where for a positive integer k the number $z(k)$ denotes the smallest positive integer m such that $k \mid F_m$, and $\mu(k)$ is the Möbius function of k. The number $z(k)$ is sometimes called the *order of appearance of* k in the Fibonacci sequence. The constant Γ_f is different from 0; hence, the sequence $(f_n)_{n\ge1}$ has linear growth. Observe that the sequence $(F_n)_{n\ge1}$ has exponential growth.

For a positive integer ℓ and a real number $x>1$ we put $\log_\ell x$ for the recursively defined function $\log_1 x := \log x$ and $\log_\ell x := \max\{\log(\log_{\ell-1} x), 1\}$. For a positive integer m we write $p(m)$ for the smallest prime factor of m, with the convention that $p(1)=1$, and $\omega(m)$ and $\tau(m)$ for the number of distinct prime divisors of m and the total number of divisors of m, respectively. We write $c_1,\ c_2,\ c_3$, etc., for some computable constants which might depend on some other parameters and which are labeled increasingly throughout the paper. We use the Landau symbol O and the Vinogradov symbol $\ll$ with their usual meanings.

2. Preliminaries

Let $\alpha := (1+\sqrt{5})/2$ and $\beta := (1-\sqrt{5})/2$ be the characteristic roots of the Fibonacci sequence. Then

$$F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta} \qquad \text{for all} \quad m \geq 1.$$

Let $m = nj$ for some positive integers j and n. Put

$$F_n^{(j)} := \frac{F_{nj}}{F_n} = \frac{\alpha^{nj} - \beta^{nj}}{\alpha^n - \beta^n}.$$

Then $F_{nj} = F_j F_n^{(j)}$. Furthermore, it is well-known that if p is any prime factor dividing both F_j and $F_n^{(j)}$, then p divides both n and F_j. In particular, if the smallest prime factor of n exceeds F_j, then F_j and $F_n^{(j)}$ are coprime.

The following results appear in [7]. For a positive integer m, let $\mathcal{P}_m := \{p : z(p) = m\}$.

Lemma 2.1. *We have the estimate*

$$\sum_{p \in \mathcal{P}_m} \frac{1}{p} \ll \frac{\log m}{m}. \tag{2.1}$$

A better estimate than (2.1) is Lemma 8 in [1], which asserts that the estimate

$$\sum_{p \in \mathcal{P}_m} \frac{1}{p-1} \leq \frac{12 + 2\log\log m}{\phi(m)}$$

holds for all $m \geq 2$.

The working horse of our paper is the following result which is Lemma 8 in [7].

Lemma 2.2. *Let $j \in \mathbb{N}$, $\varepsilon > 0$, $\gamma \in (0,1)$ and $A > 0$ be all given. Then there exist infinitely many n with $p(n) > A$ such that*

$$\frac{\phi(F_n^{(j)})}{F_n^{(j)}} \in (\gamma, \gamma + \varepsilon).$$

3. The Proofs

3.1. **Proof of Theorem 1.1.** Let $\gamma \in (0,1)$ and $\varepsilon > 0$ be arbitrary. Use Lemma 2.2 with $j = 1$ in order to conclude that there is some m such that

$$\frac{\phi(F_m)}{F_m} \in \left(\gamma + \frac{\varepsilon}{2}, \gamma + \varepsilon\right). \tag{3.1}$$

Fix such a positive integer m. Choose numbers $n := m\ell$ for some positive integer ℓ. We let $y > m$ be a parameter to be made more precise later and look only at the positive integers ℓ which are free of primes $p \leq y$. The proportion of them is

$$\prod_{p \leq y} \left(1 - \frac{1}{p}\right) > \frac{c_2}{\log y}, \tag{3.2}$$

where, by Mertens' Theorem, we can take $c_2 := e^{-\gamma}/2$ provided that y is sufficiently large. Since $p(\ell) > m$, it follows that ℓ and m are coprime so that

$$\frac{\phi(F_n)}{F_n} = \frac{\phi(F_m)}{F_m} \frac{\phi(F_\ell^{(m)})}{F_\ell^{(m)}}. \tag{3.3}$$

It suffices to show that there is a positive proportion of such ℓ with the property that

$$1-\frac{\varepsilon}{4}<\frac{\phi(F_\ell^{(m)})}{F_\ell^{(m)}}, \tag{3.4}$$

because then by estimates (3.1), (3.2), and (3.4), it will follow easily that there exists a positive proportion of n such that

$$\frac{\phi(F_n)}{F_n}\in(\gamma,\gamma+\varepsilon)\,,$$

showing that in fact $D_F(\gamma+\varepsilon)>D_F(\gamma)$. Since this is true for arbitrary $\gamma\in(0,1)$ and $\varepsilon>0$, we conclude the proof of this theorem.

To construct the positive proportion of positive integers ℓ with the desired property (3.4), we proceed as follows. Note first that

$$\begin{aligned}\frac{\phi(F_\ell^{(m)})}{F_\ell^{(m)}} &= \prod_{\substack{z(p)\mid m\ell\\ z(p)\nmid m}}\left(1-\frac{1}{p}\right)=\prod_{\substack{d_1\mid \ell\\ d_1>1}}\prod_{d_2\mid m}\prod_{z(p)=d_1d_2}\left(1-\frac{1}{p}\right)\\ &=\exp\left(\sum_{\substack{d_1\mid\ell\\ d_1>1}}\sum_{d_2\mid m}\sum_{p\in\mathcal{P}_{d_1d_2}}\frac{1}{p}+O\left(\sum_{z(p)>y}\frac{1}{p^2}\right)\right)\\ &>\exp\left(-c_3\sum_{\substack{d_1\mid\ell\\ d_1>1}}\sum_{d_2\mid m}\frac{\log(d_1d_2)}{d_1d_2}+O\left(\frac{1}{y}\right)\right).\end{aligned} \tag{3.5}$$

In the above estimates, we used Lemma 2.1 together with the fact that since $p\equiv\pm1 \pmod{z(p)}$ holds for all primes $p\neq5$, then if $z(p)=d_1d_2$ for some divisor $d_1>1$ of ℓ and some divisor d_2 of m, it follows that $d_1>y$, so $p\ge z(p)-1\ge d_1-1>y-1$. Let c_4 be the implicit constant in the Landau symbol in (3.5). Since $d_1>y>m\ge d_2$, we have

$$\frac{\log(d_1d_2)}{d_1d_2}\le\frac{\log(d_1^2)}{d_1}=\frac{2\log(d_1)}{d_1},$$

so

$$\sum_{\substack{d_1\mid\ell\\ d_1>1}}\sum_{d_2\mid m}\frac{\log(d_1d_2)}{d_1d_2}\le 2\tau(m)\sum_{\substack{d\mid\ell\\ d>1}}\frac{\log d}{d}.$$

Observing that $\log 1=0$, we get with $c_5:=2\tau(m)c_3+c_4$ and

$$S_\ell:=\sum_{d\mid\ell}\frac{\log d}{d},$$

that the inequality

$$\frac{\phi(F_\ell^{(m)})}{F_\ell^{(m)}}>\exp(-c_5S_\ell)\qquad\text{holds.}$$

Now observe that

$$\sum_{\ell\le x} S_\ell = \sum_{\ell\le x}\sum_{d\mid \ell}\frac{\log d}{d} = \sum_{\substack{p(d)>y\\ d\le x}}\frac{\log d}{d}\sum_{\substack{\ell\le x\\ \ell\equiv 0 \pmod d}} 1$$

$$= \sum_{\substack{p(d)>y\\ d\le x}}\frac{\log d}{d}\left\lfloor\frac{x}{d}\right\rfloor \le x\sum_{d>y}\frac{\log d}{d^2} \le x\int_{y-1}^{\infty}\frac{(\log t)dt}{t^2} < \frac{c_6 x\log y}{y}.$$

Thus, a first moment argument shows that the set of $\ell \le x$ such that $S_\ell \ge 1/\sqrt{y}$ is of cardinality $< c_6(\log y)/\sqrt{y}$. If y is so large that

$$\frac{c_2}{\log y} - \frac{c_6(\log y)}{\sqrt{y}} > 0,$$

it then follows, by estimate (3.2), that a positive proportion of our ℓ have the property that $S_\ell < 1/\sqrt{y}$. For such ℓ, we have

$$\frac{\phi(F_\ell^{(m)})}{F_\ell^{(m)}} > \exp(-c_5 S_\ell) > 1 - c_5 S_\ell > 1 - \frac{c_5}{\sqrt{y}} > 1 - \frac{\varepsilon}{4}$$

provided also that $y > (4c_5/\varepsilon)^2$. This is what we wanted to prove.

4. The Proof of Theorem 1.2

Here we proceed as in [3], or as in Section 5 of [7]. We first let $k > 1$ be an integer. Then we choose $\varepsilon > 0$ small enough with respect to k in a way that will be made more precise later. Put $K := k!^2$. We let n be such that $n \equiv 0 \pmod K$. Write $n := K\ell$. Now substitute for $j = 1, \ldots, k$,

$$n + j = K\ell + j = j\left(\frac{K}{j}\ell + 1\right) =: jn_j, \qquad \text{for} \quad j = 1, \ldots, k. \tag{4.1}$$

Assume that the smallest prime factor of $n_1 n_2, \ldots, n_k$ is $> F_k$. Then, $F_{n+j} = F_j F_{n_j}^{(j)}$ and the two factors on the right are coprime. Hence,

$$\frac{\phi(F_{n+j})}{F_{n+j}} = \frac{\phi(F_j)}{F_j}\frac{\phi(F_{n_j}^{(j)})}{F_{n_j}^{(j)}}.$$

For simplicity, write $C_j := \phi(F_j)/F_j$. We want to choose n such that the inclusion

$$\frac{\phi(F_{n+j})}{F_{n+j}} \in \left(\frac{\varepsilon}{2}, \varepsilon\right)$$

holds. This is equivalent to

$$\frac{\phi(F_{n_j}^{(j)})}{F_{n_j}^{(j)}} \in \left(\frac{\varepsilon}{2C_j}, \frac{\varepsilon}{C_j}\right). \tag{4.2}$$

But C_j is sub-unitary, so the condition we need is that ε is sufficiently small. Clearly, $C_1 = C_2 = 1$ and $C_3 = 1/2$. Since the inequality

$$\frac{\phi(m)}{m} \ge \frac{c_7}{\log\log m}$$

holds for all $m > 3$ with some positive constant c_7, it follows that if $j \geq 4$ (hence, $F_j \geq 3$), then

$$C_j = \frac{\phi(F_j)}{F_j} \geq \frac{c_7}{\log\log F_j} > \frac{c_7}{\log j} \geq \frac{c_7}{\log k},$$

holds for all $4 \leq j \leq k$, where we used the fact that the inequality $F_j < e^j$ holds for all $j \geq 1$. Now taking $\varepsilon := 1/(\log k)^2$, we then get that

$$\frac{\varepsilon}{C_j} \leq \frac{1}{c_7 \log k}$$

and the quantity on the right above is < 1 for all sufficiently large k.

Putting $\gamma_j := 2\varepsilon/(3C_j)$ and taking $\varepsilon_1 := \min\{\varepsilon/(12C_j)\}$, Lemma 2.2 applied with ε replaced by ε_1 successively with $j = 1, \ldots, k$ implies that there exist $m_1, \ldots, m_k$ which are pairwise coprime and such that the smallest prime factor of $m_1 m_2 \cdots m_k$ exceeds F_k and such that furthermore

$$\frac{\phi(F^{(j)}_{m_j})}{F^{(j)}_{m_j}} \in (\gamma_j, \gamma_j + \varepsilon_1) \subseteq \left(\frac{2\varepsilon}{3C_j}, \frac{\varepsilon}{C_j}\right) \qquad \text{for} \quad j = 1, \ldots, k. \tag{4.3}$$

Indeed, for $j := 1$, we choose $\gamma := \gamma_1$ and $A := F_k$ and invoke Lemma 2.2 with ε replaced by ε_1 to find m_1 such that containment (4.3) holds with $j = 1$. Next, we choose $j := 2$, $A := F_k m_1$, and $\gamma := \gamma_2$ and apply again Lemma 2.2 with ε replaced by ε_1 to get m_2 whose smallest prime factor exceeds both F_k and m_1 such that containment (4.3) holds for $j = 2$. So m_2 is coprime to m_1. Inductively, we construct $m_1, \ldots, m_k$ with the desired distributional and arithmetic properties. Now to get to n satisfying (4.2), we first use the Chinese Remainder Lemma to solve $n_j \equiv 0 \pmod{m_j}$, which is possible since $m_1, \ldots, m_k$ are coprime in pairs. This puts ℓ into an arithmetic progression N modulo $M := m_1 \ldots m_k$. Writing $\ell := N + M\lambda$, we get that

$$(n+1)(n+2)\cdots(n+k) = k! m_1 m_2 \cdots m_k Q(\lambda),$$

where $Q(X) \in \mathbb{Z}[X]$ is a linear polynomial with simple roots which factors in linear factors over the integers. By the sieve, there exist constants c_8 and c_9 such that for large x, there are $> c_7 x/(\log x)^k$ values of $\lambda \leq x$ with the property that the smallest prime factor of $Q(\lambda)$ exceeds x^{c_9}. Thus, if we write $n_j := m_j \ell_j$, then $\omega(\ell_j) < c_{10}$. For large x (say, so large that $x^{c_9} > F_{km_j}$), we have

$$\frac{\phi(F^{(j)}_{n_j})}{F^{(j)}_{n_j}} = \frac{\phi(F^{(j)}_{m_j})}{F^{(j)}_{m_j}} \frac{\phi(F^{(jm_j)}_{\ell_j})}{F^{(jm_j)}_{\ell_j}}.$$

The primes dividing $F^{(jm_j)}_{\ell_j}$ all have indices of appearance divisible by a prime factor of ℓ_j, which is larger than x^{c_9}. Thus, by Lemma 2.1, we get easily via an argument similar to the one used in the proof of Theorem 1.1, that

$$\frac{\phi(F^{(jm_j)}_{\ell_j})}{F^{(jm_j)}_{\ell_j}} = \exp\left(O\left(\frac{\log x}{x^{c_9}}\right)\right), \tag{4.4}$$

where the constant implied by the last O symbol above depends on jm_j (in fact, for large x it depends only on $\tau(jm_j)$). At any rate, as x tends to infinity, the right–hand side in estimate (4.4) above tends to 1, so if x is sufficiently large then this quantity will become larger than

$1-\varepsilon_1$. With estimate (4.3), we then get that

$$\frac{\phi(F_{n_j}^{(j)})}{F_{n_j}^{(j)}} \in (\gamma_1-\varepsilon_1, \gamma_1+\varepsilon_1) \subseteq \left(\frac{\varepsilon}{2C_j}, \frac{\varepsilon}{C_j}\right),$$

which is exactly the containment (4.2) for j.

Now having done the hardest part, the proof is not that far off. Namely, it is clear that with a large n satisfying the containments (4.2) for $j = 1, \ldots, k$, we have

$$f_{n+j} - f_{n+j-1} = \frac{\phi(F_{n+j})}{F_{n+j}} = c_j \frac{\phi(F_{n_j}^{(j)})}{F_{n_j}^{(j)}} \in (0, \varepsilon),$$

while the sum of the above numbers for $j = 1, \ldots, k$ is

$$f_{n+k} - f_n = \sum_{j=1}^{k} \frac{\phi(F_{n+j})}{F_{n+j}} > \frac{k\varepsilon}{2} = \frac{k}{2(\log k)^2} > 1,$$

for k sufficiently large, from where one deduces easily that $(f_n)_{n\ge1}$ is indeed dense modulo 1.

5. Concluding Remarks

The method of proofs of both results extends easily to other sequences. For example, it extends to the function $\nu(F_n)$, where ν is a multiplicative function, not necessarily strongly multiplicative, with values in $(0, 1]$ such that for some positive constant c we have

$$\nu(p^a) = 1 - \frac{c}{p} + O\left(\frac{1}{p^2}\right) \quad \text{for all primes} \quad p \ge 2 \quad \text{and integers} \quad a \ge 1.$$

For example, we can replace $\phi(F_m)/F_m$ with $F_m/\sigma(F_m)$, where $\sigma(m)$ is the sum of the divisors of m, and keep the conclusions about the monotonicity of the distribution function and the density modulo 1 of the summatory function. It also applies when the sequence of Fibonacci numbers $(F_m)_{m\ge1}$ is replaced with the sequence of Mersenne numbers $(2^m - 1)_{m\ge1}$, or some other Lucas sequences satisfying some mild technical conditions.

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IMB, Université de Bordeaux et CNRS, 33405 Talence cedex, France
E-mail address: jean-marc.deshouillers@math.u-bordeaux.fr

Instituto de Matemáticas, Universidad Nacional Autónoma de México, C.P. 58089, Morelia, Michoacán, México
E-mail address: fluca@matmor.unam.mx