FACTORIZATION OF LENS SEQUENCES

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Abstract. We show that integral lens sequences can be factorized, thereby proving a conjecture of J. Kocik.

1. Introduction

Kocik [1] introduced the idea of a lens sequence. Two congruent circles overlap, thus forming a lens. With the common chord as axis, one can inscribe a sequence of circles in the lens centered on the axis, each tangent to the previous circle in the sequence and to the two original circles. Each of the circles in the sequence has a curvature (the reciprocal of its radius), b_n, and $\{b_n\}$ is the sequence appropriately called a lens sequence.

A lens sequence is given by $b_0 = a$, $b_1 = b$, $b_2 = c$ and, for all n,

$$b_{n-1} - \alpha b_n + b_{n+1} + \beta = 0, \tag{1.1}$$

where

$$\alpha = \frac{ab+bc+ca}{b^2} - 1 \quad \text{and} \quad \beta = \frac{ac-b^2}{b}.$$

The most interesting case occurs when a, b, c, α and β are all integers, for then $\{b_n\}$ is a bilateral sequence of integers. We give two examples:

(1) $(a, b, c, \alpha, \beta) = (2, 3, 6, 3, 1)$

$$b_{n-1} - 3b_n + b_{n+1} + 1 = 0,$$
$$\{b_n\} = \{\ \cdots, 6, 3, 2, 2, 3, 6, 14, 35, 90,\ \cdots\ \}$$

and

(2) $(a, b, c, \alpha, \beta) = (12, 20, 55, 4, 13)$,

$$b_{n-1} - 4b_n + b_{n+1} + 13 = 0,$$
$$\{b_n\} = \{\ \cdots, 112, 35, 15, 12, 20, 55, 187, 680, 2520,\ \cdots\ \}.$$

Kocik noticed that in these, and indeed in every example he studied, the lens sequence can be factorized in the following way.

In example (1),

$$\{b_n\} = \{\ \cdots, 2\times 3, 3\times 1, 1\times 2, 2\times 1, 1\times 3, 3\times 2, 2\times 7, 7\times 5, 5\times 18,\ \cdots\ \},$$
$$b_n = f_n f_{n+1} \quad \text{where} \quad \{f_n\} = \{\ \cdots\ , 2, 3, 1, 2, 1, 3, 2, 7, 5, 18,\ \cdots\ \}$$

and $\{f_n\}$ satisfies the homogeneous recurrences

$$f_{n-1} - f_n + f_{n+1} = 0, \quad \text{if } n \text{ even},$$
$$f_{n-1} - 5f_n + f_{n+1} = 0, \quad \text{if } n \text{ odd},$$

while in example (2),

$$\{b_n\} = \{\ \cdots, 16\times 7, 7\times 5, 5\times 3, 3\times 4, 4\times 5, 5\times 11, 11\times 17, 17\times 40, 40\times 63,\ \cdots\},$$

$$b_n = f_n f_{n+1} \text{ where } \{f_n\} = \{\ \cdots, 16.7, 5, 3, 4, 5, 11, 17, 40, 63,\ \cdots\}$$

and

$$\begin{aligned} f_{n-1} - 3f_n + f_{n+1} &= 0 \text{ if } n \text{ even},\\ f_{n-1} - 2f_n + f_{n+1} &= 0 \text{ if } n \text{ odd}. \end{aligned}$$

We shall prove the following theorem.

Theorem 1.1. *For an integer lens sequence (as defined above) we have*

$$b_n = f_n f_{n+1}$$

where $\{f_n\}$ *is an integer sequence satisfying*

$$\begin{aligned} f_{n-1} - \frac{f_1+f_3}{f_2} f_n + f_{n+1} &= 0 \quad \textit{if } n \textit{ even},\\ f_{n-1} - \frac{f_0+f_2}{f_1} f_n + f_{n+1} &= 0 \quad \textit{if } n \textit{ odd}. \end{aligned} \tag{1.2}$$

2. The Proof

Proof. Given that b and $\beta = \dfrac{ac-b^2}{b}$ are integers, it follows that $\dfrac{ac}{b}$ is an integer. We can write $b = d_1 d_2$ where $d_1|a$, $d_2|c$.

Let $f_0 = \dfrac{a}{d_1}$, $f_1 = d_1$, $f_2 = d_2$, $f_3 = \dfrac{c}{d_2}$. Then f_0, f_1, f_2, and f_3 are all integers. Define the sequence $\{f_n\}$ for all n by

$$f_{n-2} - \alpha f_n + f_{n+2} = 0.$$

Then $\{f_n\}$ is a sequence of integers.

We solve this recurrence explicitly. First suppose $\alpha \neq 2$. The characteristic polynomial of the recurrence is

$$x^4 - \alpha x^2 + 1 = (x^2-\lambda)(x^2-\mu) = (x-\gamma)(x+\gamma)(x-\delta)(x+\delta)$$

where

$$\lambda = \frac{\alpha+\sqrt{\alpha^2-4}}{2}, \qquad \mu = \frac{\alpha-\sqrt{\alpha^2-4}}{2},$$
$$\gamma = \frac{\sqrt{\alpha+2}+\sqrt{\alpha-2}}{2}, \quad \delta = \frac{\sqrt{\alpha+2}-\sqrt{\alpha-2}}{2}.$$

It follows that

$$f_n = A\gamma^n + B\delta^n + C(-\gamma)^n + D(-\delta)^n, \tag{2.1}$$

where A, B, C, and D are determined from

$$\begin{aligned} A + B + C + D &= f_0,\\ A\gamma + B\delta - C\gamma - D\delta &= f_1,\\ A\gamma^2 + B\delta^2 + C\gamma^2 + D\delta^2 &= f_2,\\ A\gamma^3 + B\delta^3 - C\gamma^3 - D\delta^3 &= f_3. \end{aligned}$$

We find that

$$\begin{aligned} A &= \frac{1}{2(\gamma^2-\delta^2)}\left(-\delta^2 f_0 - \delta^3 f_1 + f_2 + \delta f_3\right), \\ B &= \frac{1}{2(\gamma^2-\delta^2)}\left(\gamma^2 f_0 + \gamma^3 f_1 - f_2 - \gamma f_3\right), \\ C &= \frac{1}{2(\gamma^2-\delta^2)}\left(-\delta^2 f_0 + \delta^3 f_1 + f_2 - \delta f_3\right), \\ D &= \frac{1}{2(\gamma^2-\delta^2)}\left(\gamma^2 f_0 - \gamma^3 f_1 - f_2 + \gamma f_3\right). \end{aligned} \tag{2.2}$$

We now calculate $f_n f_{n+1}$.

$$\begin{aligned} f_n f_{n+1} &= (A\gamma^n + B\delta^n + C(-\gamma)^n + D(-\delta)^n) \\ &\quad \times (A\gamma^{n+1} + B\delta^{n+1} + C(-\gamma)^{n+1} + D(-\delta)^{n+1}) \\ &= (A^2 - C^2)\gamma^{2n+1} + (B^2 - D^2)\delta^{2n+1} + (AB - CD)(\gamma + \delta) \\ &\quad + (AD - BC)(\gamma - \delta)(-1)^n \\ &= (A^2 - C^2)\gamma\lambda^n + (B^2 - D^2)\delta\mu^n + (AB - CD)(\gamma + \delta) \\ &\quad + (AD - BC)(\gamma - \delta)(-1)^n \end{aligned} \tag{2.3}$$

where we have used the fact that $\gamma^2 = \lambda$, $\delta^2 = \mu$, and $\gamma\delta = 1$.

On the other hand, the solution to the non-homogeneous equation for b_n is easily found to be

$$\begin{aligned} b_n &= a\left(\frac{\lambda\mu^n - \mu\lambda^n}{\lambda - \mu}\right) + b\left(\frac{\lambda^n - \mu^n}{\lambda - \mu}\right) - \frac{\beta}{\alpha - 2}\left(\frac{\lambda\mu^n - \mu\lambda^n}{\lambda - \mu} + \frac{\lambda^n - \mu^n}{\lambda - \mu} - 1\right) \\ &= \left(a - \frac{\beta}{\alpha - 2}\right)\left(\frac{\lambda\mu^n - \mu\lambda^n}{\lambda - \mu}\right) + \left(b - \frac{\beta}{\alpha - 2}\right)\left(\frac{\lambda^n - \mu^n}{\lambda - \mu}\right) + \frac{\beta}{\alpha - 2} \\ &= \frac{1}{\lambda - \mu}\left(-\mu a + b + (\mu - 1)\frac{\beta}{\alpha - 2}\right)\lambda^n + \frac{1}{\lambda - \mu}\left(\lambda a - b + (-\lambda + 1)\frac{\beta}{\alpha - 2}\right)\mu^n + \frac{\beta}{\alpha - 2}. \end{aligned} \tag{2.4}$$

From (2.2) it follows routinely that

$$\begin{aligned} AD - BC &= 0, \\ (AB - CD)(\gamma + \delta) &= \frac{\beta}{\alpha - 2}, \\ (A^2 - C^2)\gamma &= \frac{1}{\lambda - \mu}\left(-\mu a + b + (\mu - 1)\frac{\beta}{\alpha - 2}\right), \\ (B^2 - D^2)\delta &= \frac{1}{\lambda - \mu}\left(\lambda a - b + (-\lambda + 1)\frac{\beta}{\alpha - 2}\right), \end{aligned} \tag{2.5}$$

(the details are left to the reader) and hence,

$$b_n = f_n f_{n+1}.$$

Next, from (2.1) we have

$$\begin{aligned} f_{2n+1} &= (A-C)\gamma^{2n+1} + (B-D)\delta^{2n+1}, \\ f_{2n} &= (A+C)\gamma^{2n} + (B+D)\delta^{2n}, \\ f_{2n-1} &= (A-C)\gamma^{2n-1} + (B-D)\delta^{2n-1}. \end{aligned} \tag{2.6}$$

Again from (2.2) it follows routinely that

$$\begin{aligned} (A-C)(\gamma+\delta) &= \frac{f_1+f_3}{f_2}(A+C), \\ (B-D)(\gamma+\delta) &= \frac{f_1+f_3}{f_2}(B+D), \end{aligned}$$

and hence,

$$f_{2n-1} + f_{2n+1} = \frac{f_1+f_3}{f_2} f_{2n}. \tag{2.7}$$

Similarly from (2.1),

$$\begin{aligned} f_{2n+2} &= (A+C)\gamma^{2n+2} + (B+D)\delta^{2n+2}, \\ f_{2n+1} &= (A-C)\gamma^{2n+1} + (B-D)\delta^{2n+1}, \\ f_{2n} &= (A+C)\gamma^{2n} + (B+D)\delta^{2n}, \end{aligned} \tag{2.8}$$

it follows from (2.2) that

$$\begin{aligned} (A+C)(\gamma+\delta) &= \frac{f_0+f_2}{f_1}(A-C), \\ (B+D)(\gamma+\delta) &= \frac{f_0+f_2}{f_1}(B-D) \end{aligned} \tag{2.9}$$

and hence,

$$f_{2n} + f_{2n+2} = \frac{f_0+f_2}{f_1} f_{2n+1}, \tag{2.10}$$

and the proof is complete in the case $\alpha \neq 2$.

Now suppose $\alpha = 2$. The characteristic polynomial of the recurrence is

$$x^4 - 2x^2 + 1 = (x-1)^2(x+1)^2,$$

and it follows that

$$f_n = (An+B) + (Cn+D)(-1)^n, \tag{2.11}$$

where A, B, C, and D are determined from

$$\begin{aligned} &B + D = f_0, \\ &A + B - C - D = f_1, \\ &2A + B + 2C + D = f_2, \\ &3A + B - 3C - D = f_3. \end{aligned}$$

We find that

$$A = \frac{1}{4}(-f_0 - f_1 + f_2 + f_3), \tag{2.12}$$
$$B = \frac{1}{4}(2f_0 + 3f_1 - f_3),$$
$$C = \frac{1}{4}(-f_0 + f_1 + f_2 - f_3),$$
$$D = \frac{1}{4}(2f_0 - 3f_1 + f_3),$$

and

$$\begin{aligned} f_n f_{n+1} &= (An + B + Cn(-1)^n + D(-1)^n) \\ &\quad \times (An + (A+B) + Cn(-1)^n + (C+D)(-1)^n) \\ &= (A^2 - C^2)n^2 + (A^2 + 2AB - C^2 - 2CD)n \\ &\quad + (AB + B^2 - CD - D^2) + (AD - BC)(-1)^n. \end{aligned} \tag{2.13}$$

On the other hand,

$$b_n = -\frac{\beta}{2}n^2 + \left(-a + b + \frac{\beta}{2}\right)n + a. \tag{2.14}$$

From (2.12) it follows that

$$AD - BC = 0, \tag{2.15}$$
$$AB + B^2 - CD - D^2 = a,$$
$$A^2 - C^2 = -\frac{\beta}{2},$$
$$AB - CD = \frac{1}{2}(-a + b + \beta)$$

and hence,

$$b_n = f_n f_{n+1}.$$

Next, from (2.11) we have

$$f_{2n+1} = (2A - 2C)n + (A + B - C - D), \tag{2.16}$$
$$f_{2n} = (2A + 2C)n + (B + D),$$
$$f_{2n-1} = (2A - 2C)n + (-A + B + C - D).$$

From (2.12) it follows that

$$4A - 4C = \frac{f_1 + f_3}{f_2}(2A + 2C),$$
$$2B - 2D = \frac{f_1 + f_3}{f_2}(B + D),$$

and hence,

$$f_{2n-1} + f_{2n+1} = \frac{f_1 + f_3}{f_2} f_{2n}. \tag{2.17}$$

Similarly,

$$f_{2n+2} = (2A + 2C)n + (2A + B + 2C + D), \tag{2.18}$$
$$f_{2n+1} = (2A - 2C)n + (A + B - C - D),$$
$$f_{2n} = (2A + 2C)n + (B + D),$$

from (2.12),

$$4A + 4C = \frac{f_0 + f_2}{f_1}(2A - 2C), \tag{2.19}$$
$$2A + 2B + 2C + 2D = \frac{f_0 + f_2}{f_1}(A + B - C - D),$$

and

$$f_{2n} + f_{2n+2} = \frac{f_0 + f_2}{f_1} f_{2n+1}, \tag{2.20}$$

and the proof is complete. □

References

[1] J. Kocik, *Lens sequences*, arXiv:0710.3226v1 [math.NT]

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