RESULTS ON THE $3x + 1$ AND $3x + d$ CONJECTURES

DHIRAJ HOLDEN

Abstract. We give results relating to the $3x+1$ and $3x+d$ conjectures, proposed by Collatz and Lagarias, respectively. We prove two theorems about the Primitive Cycles Existence Conjecture, which give a sufficient condition for which a primitive cycle will exist for a positive integer d, and list the first few primitive cycles found using this condition.

1. Introduction

The Collatz function is defined as

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \pmod 2 \\ \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod 2 \end{cases} \tag{1.1}$$

for positive integers x [2]. The Collatz conjecture states that, for all positive integers n, $T^k(n) = 1$ for some integer k. This conjecture remains unsolved, despite repeated attempts to solve it. The function was generalized by Lagarias to

$$T_d(x) = \begin{cases} \frac{x}{2} & \text{if } x \equiv 0 \pmod 2 \\ \frac{3x+d}{2} & \text{if } x \equiv 1 \pmod 2 \end{cases} \tag{1.2}$$

for d relatively prime to 6.

Lagarias developed a generalization of the $3x + 1$ conjecture using the generalized Collatz function. He stated two conjectures about the $3x + d$ function relating to cycles. A cycle is defined as a sequence of numbers $n, T_d(n), T_d^2(n), T_d^3(n), \ldots, T_d^k(n)$ for a positive integer n and a positive integer d relatively prime to 6 such that $T_d^k(n) = n$. A primitive cycle meets all of these conditions with the additional requirement that n is relatively prime to d. Lagarias states the Primitive Cycles Existence Conjecture, which states that for every positive integer d relatively prime to 6, there exists at least one primitive cycle for $T_d(x)$, and the Finite Primitive Cycles Conjecture, which states that the number of such cycles is finite for any such d [3]. Also, Simons, in [4], proves upper and lower bounds for the number of primitive cycles of a given length. This paper is a continuation of the work of Belaga and Mignotte, as shown below.

Definition 1.1. (Belaga [1]) A Collatz number is a positive integer of the form $2^j - 3^k$, where j and k are positive integers.

Definition 1.2. (Belaga [1]) The Collatz corona for a Collatz number $2^j - 3^k$ as defined by Belaga and Mignotte is the set of integers of the form

$$3^{k-1} + 3^{k-2}2^{e_1} + 3^{k-3}2^{e_1+e_2} + \cdots + 2^{e_1+e_2+\cdots+e_{k-1}}$$

for an aperiodic sequence of positive integers $e_1, e_2, \ldots, e_k$ so that

$$e_1 + e_2 + \cdots + e_k = j.$$

Theorem 1.3 ([1])**.** *$n = T^k(n)$ if $n = \frac{3^k n + A*d}{2^j}$, $(2^j - 3^k)n = A*d$, and $B*n = A*d$ for some Collatz number B and a number A in the Collatz corona for B.*

The paper proves theorems which are used to state conditions under which a primitive cycle can occur. The two theorems below are new results, and the proofs are given in Section 2.

Theorem 1.4. *Let q, k, and j be positive integers such that q is relatively prime to 6, 2 is a primitive root $\mod q$, $2^j - 3^k$ is positive, $k - 2 + \phi(q) < j$, and $3^{k-1} - 2^{k-1}$ is relatively prime to q. Then there exist positive integers $e_1, e_2, \dots, e_{k-1}$ such that q divides*

$$n = 3^{k-1} + 3^{k-2}2^{e_1} + 3^{k-3}2^{e_1+e_2} + \cdots + 2^{e_1+e_2+\cdots+e_{k-1}},$$

and n is in the Collatz corona of $2^j - 3^k$.

Theorem 1.5. *Let B be a Collatz number of the form $2^j - 3^k$ which is divisible by some d relatively prime to 6. If 2 is a primitive root of $\frac{B}{d}$ $k - 2 + \phi\left(\frac{B}{d}\right) < j$, $3^{k-1} - 2^{k-1}$ is relatively prime to $\frac{B}{d}$, and if*

$$x = \frac{(3(3^{k-1} - 2^{k-1}) + 2^{k-2+e_{k-1}})d}{(2^j - 3^k)}$$

is relatively prime to d, where e_{k-1} is the smallest positive value that makes x as defined above divisible by $\frac{B}{d}$, then there exists a primitive cycle for d.

2. Proofs

Proof of Theorem 1.4. Choose $e_1, e_2, \dots, e_{k-1}$ such that $e_1 = e_2 = e_3 \cdots = e_{k-2} = 1$. Then let x be defined by the least positive residue of $3^{k-1} + 3^{k-2}2^{e_1} + \cdots + (3)2^{e_1+e_2+\cdots+e_{k-2}} \pmod q$. If $3^{k-1} - 2^{k-1}$ is relatively prime to q, x is relatively prime to q because $x \equiv 3^{k-1} + 3^{k-2}2^{e_1} + \cdots + (3)2^{e_1+e_2+\cdots+e_{k-2}} = 3(3^{k-2} + 3^{k-3}2^1 + 3^{k-4}2^2 + \cdots + 2^{k-2}) = 3(3^{k-1} - 2^{k-1})$ since $e_1 = e_2 = e_3 \cdots = e_{k-2} = 1$. Therefore, there exist an infinite number of numbers e_{k-1} such that $2^{e_1+e_2+\cdots+e_{k-1}} \equiv -x \pmod q$ since 2 is a primitive root of q. Since the order of 2 mod $q = \phi(q)$ as 2 is a primitive root of q, the lowest value of e_{k-1} such that $n \equiv 0 \pmod q$, where n is defined as above, is at most $\phi(q)$. If $k - 2 + \phi(q) < j$, then n is in the Collatz corona of $2^j - 3^k$ by the definition of the Collatz corona, since $e_k = j - (e_1 + e_2 + \cdots + e_{k-1})$. □

Note that the above theorem is not true for arbitrary q, j, and k because if $k - 2 + \phi(q)$ is not less than j, or if 2 is not a primitive root $\pmod q$, there is no guarantee then that e_{k-1} can be chosen such that q divides $3^{k-1} + 3^{k-2}2^{e_1} + 3^{k-3}2^{e_1+e_2} + \cdots + 2^{e_1+e_2+\cdots+e_{k-1}}$.

Proof of Theorem 1.5. If 2 is a primitive root of $\frac{B}{d}$, $k - 2 + \phi\left(\frac{B}{d}\right) < j$ and $3^{k-1} - 2^{k-1}$ is relatively prime to $\frac{B}{d}$, then a number in the Collatz corona of B divisible by $\frac{B}{d}$ exists by Theorem 1.4, denoted above by x. This means that $\frac{x \cdot d}{B}$ is an integer, and since $n = \frac{x \cdot d}{B}$ is a positive integer, by Theorem 1.3, there exists a primitive cycle for d if n is relatively prime to d. If x defined above is relatively prime to d, n is relatively prime to d and a primitive cycle exists for d. □

3. Table of Primitive Cycle Values for Small d

The following table gives values of primitive cycles that are generated by numbers that satisfy the conditions of Theorem 1.5. Theorem 1.5 gives a sufficient but not necessary condition for a primitive cycle to exist for a number d. Since Theorem 1.5 depends on the conditions that there exists a Collatz number B such that 2 is a primitive root of $\frac{B}{d}$, that $k - 2 + \phi\left(\frac{B}{d}\right)$ is less than j, and that $3^{k-1} - 2^{k-1}$ is relatively prime to $\frac{B}{d}$, not all numbers satisfy this condition and thus not all values for primitive cycles are listed in the table. There exist other primitive cycles for some d, such as the cycle starting with 187 for the $3x + 5$ map [4].

d	$B/d = q$ in Theorem 1.4	j	k	Primitive cycle values[4]
1	1	2	1	1 2
5	1	3	1	1 4 2
	1	5	3	19 31 49 76 38
	1	5	3	23 37 58 29 46
7	1	4	2	5 11 20 10
11	5	6	2	1 7 16 8 4 2
13	1	4	1	1 8 4 2
	1	8	5	211 323 491 743 1121 1688 844 422
	1	8	5	259 395 599 905 1364 682 341 518
	1	8	5	227 347 527 797 1202 601 908 454

TABLE 1. Primitive cycles given by Theorem 1.5 for small d.

4. Acknowledgements

I would like to acknowledge Dr. Penny Haxell, University of Waterloo, Dr. Jeffrey Lagarias, University of Michigan, and Dr. Oscar Vega, California State University, Fresno, for their help in reviewing this paper and their suggestions. I would also like to acknowledge the anonymous referee for his comments on this work.

References

[1] E. G. Belaga and M. Mignotte, *Walking cautiously into the Collatz wilderness: algorithmically, number theoretically, randomly*, in DMTCS Proceedings, North America, 2006.

[2] J. C. Lagarias, *The 3x+1 problem and its generalizations*, The American Mathematical Monthly, **92** (1985), 3–23.

[3] J. C. Lagarias, *The set of rational cycles for the 3x+1 problem*, Acta Arithmetica, **56** (1990), 33–53.

[4] J. L. Simons, *On the (non)-existence of m-cycles for generalized Syracuse sequences*, Acta Arithmetica, **131** (2008), 217–254.

MSC2010: 11D61

California State University, Fresno, Fresno, California 93740

University High School, Fresno, California 93740,
E-mail address: dhiraj.holden@gmail.com