ON THE NUMBER OF SUMMANDS IN
ZECKENDORF DECOMPOSITIONS

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ABSTRACT. Zeckendorf proved that every positive integer has a unique representation as a sum of non-consecutive Fibonacci numbers. Once this has been shown, it's natural to ask how many summands are needed. Using a continued fraction approach, Lekkerkerker proved that the average number of such summands needed for integers in $[F_n, F_{n+1})$ is $n/(\alpha^2 + 1) + O(1)$, where F_n is the nth Fibonacci number and $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden mean. Surprisingly, no one appears to have investigated the distribution of the number of summands; our main result is that this converges to a Gaussian as $n \to \infty$. Moreover, such a result holds not just for the Fibonacci numbers but many other problems, such as linear recurrence relations with non-negative integer coefficients (which is a generalization of base B expansions of numbers) and far-difference representations.

In general, the proofs involve adopting a combinatorial viewpoint and analyzing the resulting generating functions through partial fraction expansions and differentiating identities. The resulting arguments become quite technical. The purpose of this paper is to concentrate on the special and most interesting case of the Fibonacci numbers, where the obstructions vanish and the proofs follow from some combinatorics and Stirling's formula; see [13] for proofs in the general case.

1. INTRODUCTION

Fibonacci numbers are one of the most well-known and studied sequences in mathematics, as well as one of the most enjoyable to play with. There are books, journals and conferences dedicated to all their wondrous properties. The purpose of this article is to review two nice results, namely Zeckendorf's and Lekkerkerker's Theorems, and discuss generalizations and related questions. We provide complete proofs for the Fibonacci sequence and refer the reader to [13] for proofs of the general cases.

Before stating our results, we first set some notation. We study the (shifted) Fibonacci sequence given by $\mathcal{F}_1 = 1$, $\mathcal{F}_2 = 2$, $\mathcal{F}_3 = 3$, $\mathcal{F}_4 = 5$ and in general $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2}$. Note our sequence is offset from the standard Fibonacci sequence $\{F_n\}$ by 1, with $\mathcal{F}_n = F_{n+1}$. The nth term in the sequence is given by the (shifted) Binet's formula

$$\mathcal{F}_n \;=\; \frac{\alpha}{\sqrt{5}} \cdot \alpha^n - \frac{1-\alpha}{\sqrt{n}} \cdot (1-\alpha)^n, \quad \alpha \;=\; \frac{1+\sqrt{5}}{2}, \tag{1.1}$$

where α is the golden mean. We will see below why it is convenient to study the shifted sequence. We refer to the elements $\{\mathcal{F}_n\}$ as (shifted) Fibonacci numbers.

Zeckendorf (see for instance [16]) proved that every positive integer can be written uniquely as a sum of non-adjacent (shifted) Fibonacci numbers. The proof is a straightforward induction. Note, though, how important it is that our series begins with just a single 1; if we had two 1's then the decompositions of many numbers into non-adjacent summands would not be unique.

In the 1950's, Lekkerkerker [10] answered the following question, which is a natural outgrowth of Zeckendorf's theorem: *On average, how many summands are needed in the Zeckendorf decomposition?* Lekkerkerker proved that for integers in $[\mathcal{F}_n, \mathcal{F}_{n+1})$ the average number of summands, as $n \to \infty$, is $\frac{n}{\alpha^2+1} + O(1) \approx .276n$.

Of course, one can ask these questions for more general recurrence relations. Zeckendorf's result has been generalized to several recurrence relations (see the 1972 special volume on representations in the Fibonacci Quarterly, especially [6, 7], as well as [11]). Burger [2] proved the analogous result for the mean number of summands for a generalization of Fibonacci numbers, $G_n = G_{n-1} + G_{n-2} + \cdots + G_{n-L}$. There is, of course, another generalization, which interestingly does not seem to have been asked. Namely, *how are the number of summands distributed about the mean for integers in $[\mathcal{F}_n, \mathcal{F}_{n+1})$.* This is a very natural question to ask. Both the question and the answer are reminiscent of the Erdős-Kac Theorem [4], which states that as $n \to \infty$ the number of distinct prime divisors of integers on the order of size n tends to a Gaussian with mean $\log \log n$ and standard deviation $\sqrt{\log \log n}$. (In [13], a companion paper to this work, we show a connection between our result and the Erdős-Kac Theorem for recurrence relations of depth 1.)

Our main result is that a similar statement about Gaussian behavior holds, not just for the shifted Fibonacci numbers, but for the large class of recurrence relations (which includes the Fibonacci numbers as a special case) defined below.

Definition 1.1. *We say a sequence $\{H_n\}_{n=1}^{\infty}$ of positive integers is a Positive Linear Recurrence Sequence (PLRS) if the following properties hold:*

(1) Recurrence relation: *There are non-negative integers $L, c_1, \ldots, c_L$ such that*

$$H_{n+1} = c_1 H_n + \cdots + c_L H_{n+1-L},$$

with L, c_1 and c_L positive.

(2) Initial conditions: *$H_1 = 1$, and for $1 \leq n < L$ we have*

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1.$$

We call a decomposition $\sum_{i=1}^{m} a_i H_{m+1-i}$ of a positive integer N (and the sequence $\{a_i\}_{i=1}^{m}$) legal if $a_1 > 0$, the other $a_i \geq 0$, and one of the following two conditions hold:

- *We have $m < L$ and $a_i = c_i$ for $1 \leq i \leq m$.*
- *There exists $s \in \{0, \ldots, L\}$ such that*

$$a_1 = c_1, \quad a_2 = c_2, \quad \ldots, \quad a_{s-1} = c_{s-1} \quad \text{and} \quad a_s < c_s, \tag{1.2}$$

$a_{s+1}, \ldots, a_{s+\ell} = 0$ for some $\ell \geq 0$, and $\{b_i\}_{i=1}^{m-s-\ell}$ (with $b_i = a_{s+\ell+i}$) is legal.

If $\sum_{i=1}^{m} a_i H_{m+1-i}$ is a legal decomposition of N, we define the number of summands (of this decomposition of N) to be $a_1 + \cdots + a_m$.

Informally, a legal decomposition is one where we cannot use the recurrence relation to replace a linear combination of summands with another summand, and the coefficient of each summand is appropriately bounded. For example, if $H_{n+1} = 2H_n + 3H_{n-1} + H_{n-2}$, then $H_5 + 2H_4 + 3H_3 + H_1$ is legal, while $H_5 + 2H_4 + 3H_3 + H_2$ is not (we can replace $2H_4 + 3H_3 + H_2$

with H_5), nor is $7H_5 + 2H_2$ (as the coefficient of H_5 is too large). Note the shifted Fibonacci numbers are just the special case of $L = 2$ and $c_1 = c_2 = 1$.

We adopt a probabilistic language to state our main results.

Definition 1.2 (Associated Probability Space to a Positive Linear Recurrence Sequence). *Let* $\{H_n\}$ *be a Positive Linear Recurrence Sequence. For each n, consider the discrete outcome space*

$$\Omega_n = \{H_n, \ H_n + 1, \ H_n + 2, \ \ldots, \ H_{n+1} - 1\} \tag{1.3}$$

with probability measure

$$\mathbb{P}_n(A) = \sum_{\substack{\omega \in A \\ \omega \in \Omega_n}} \frac{1}{H_{n+1} - H_n}, \quad a \subset \Omega_n; \tag{1.4}$$

in other words, each of the $H_{n+1} - H_n$ numbers is weighted equally. We define the random variable K_n by setting $K_n(\omega)$ equal to the number of summands of $\omega \in \Omega_n$ in its legal decomposition. It is also convenient to study $\mathcal{K}_n = K_n - 1$; every integer in Ω_n must have at least one H_n as a summand; thus $\mathcal{K}_n$ is the number of non-forced summands in a legal decomposition.

Implicit in this definition is that each integer has a unique legal decomposition; we will prove this fact, and thus K_n is well-defined. Our main result is the following theorem.

Theorem 1.3. *If $\{H_n\}_{n=1}^{\infty}$ is a Positive Linear Recurrence Sequence* then every positive *integer may be written uniquely as a legal sum $\sum_i a_i H_i$ (see Definition 1.1). Let K_n be the random variable of Definition 1.2. Then $\mathbb{E}[K_n]$ and $\mathrm{Var}(K_n)$ are of order n, and as $n \to \infty$, K_n and $\mathcal{K}_n$ converge to a Gaussian.*

Our result extends Zeckendorf's and Lekkerkerker's Theorems to a large class of recurrence relations, and further implies that the distribution of summands converges to a Gaussian. The proof has three main ingredients. The first is to adopt a combinatorial point of view. Previous approaches to Lekkerkerker's Theorem were number theoretic, involving continued fractions. We instead view this as a combinatorial problem, namely how many ways can we choose elements in a set subject to some restrictions on what may be taken. This approach gives us an explicit formula for the number of N in $[H_n, H_{n+1})$ that have a given number of summands. For each interval $[H_n, H_{n+1})$ (i.e., for each n) we may thus associate a probability function p_n on $[H_n, H_{n+1})$ where $p_n(k)$ is the probability of having exactly $k+1$ summands. As remarked earlier, we choose to write the density this way as every integer in $[H_n, H_{n+1})$ must have H_n in its decomposition, and thus it is more natural to study the number of additional, non-forced summands needed. Note p_n is the density of the random variable $\mathcal{K}_n$. We then use generating functions and differentiating identities to obtain tractable formulas for these summand functions p_n, and prove our claims about the mean and the variance. We conclude by showing that as $n \to \infty$ the centered and normalized moments of p_n tend to the moments of the standard normal; by Markov's Method of Moments this yields the Gaussian behavior.

We can gain a lot of intuition as to why these results are true by looking at the special case of $L = 1$ and $c_1 = B > 0$. For these choices, our sequence is just $H_n = B^{n-1}$; in other words, we are looking at the base B decomposition of integers. Zeckendorf's Theorem is now clearly true, as every number has a unique representation. Lekkerkerker and the Gaussian behavior are now just consequences of the Central Limit Theorem. For example, consider a decomposition of an $N \in [B^n, B^{n+1})$:

$$N = a_1 B^n + a_2 B^{n-1} + \cdots + a_{n+1} B^0.$$

We have $a_1 \in \{1, \dots, B-1\}$ and all other $a_i \in \{0, \dots, B-1\}$. We are interested in the behavior, for large n, of $a_1 + \cdots + a_{n+1}$ as we vary over N in $[B^n, B^{n+1})$. Note for large n the contribution of a_1 is immaterial, and the remaining a_i's can be understood by considering the sum of n independent, identically distributed uniform random variables on $\{0, \dots, B-1\}$ (which have mean $\frac{B-1}{2}$ and standard deviation $\sqrt{(B^2-1)/12}$). Denoting these by A_i, by the Central Limit Theorem $A_2 + \cdots + A_{n+1}$ converges to a Gaussian with mean $\frac{B-1}{2}n$ and standard deviation $n\sqrt{(B^2-1)/12}$.

Our approach is quite general, and can handle a variety of related problems. We state just one more, which allows us to see some very interesting behavior.

Recently Alpert [1] showed that every positive integer can be written uniquely as a sum and difference of the shifted Fibonacci numbers $\{\mathcal{F}_n\}_{n=1}^{\infty}$ such that every two terms of the same sign differ in index by at least 4, and every two terms of opposite sign differ in index by at least 3; we call this the far-difference representation. For example,

$$2011 = \mathcal{F}_{17} - \mathcal{F}_{14} + \mathcal{F}_8 + \mathcal{F}_3 \quad \text{and} \quad 1900 = \mathcal{F}_{17} - \mathcal{F}_{14} - \mathcal{F}_{10} + \mathcal{F}_6 + \mathcal{F}_2.$$

If $S_n = \sum_{0 < n-4i \le n} \mathcal{F}_{n-4i} = \mathcal{F}_n + \mathcal{F}_{n-4} + \cdots$ for positive n and 0 otherwise, then for each $N \in (S_{n-1}, S_n]$ the first term in its far-difference representation is $\mathcal{F}_n$. Note that 0 has the empty representation. We can show the following (see [13] for the proof).

Theorem 1.4. *Consider the outcome space $\Omega_n = \{S_{n-1} + 1, S_{n-1} + 2, \dots, S_n\}$ with probability measure $\mathbb{P}_{(A)} = \sum_{\omega \in A} \frac{1}{S_n - S_{n-1}}$ for $A \subset \Omega_n$. Let K_n and L_n be the random variables denoting the number of positive and negative shifted Fibonacci summands in the far-difference representation (they are well-defined by [1]). As $n \to \infty$, for any real numbers a and b the random variable $aK_n + bL_n$ converges to a Gaussian. The expected value of K_n (which is $\alpha/2$ more than that of L_n) is $\frac{n}{10} + \frac{371-113\sqrt{5}}{40} + o(1)$; the variance of both is of size $\frac{15+21\sqrt{5}}{1000}n$. K_n and L_n are negatively correlated, with a correlation coefficient of $-(21-2\alpha)/(29+2\alpha)$. Further, $K_n + L_n$ and $K_n - L_n$ are independent random variables as $n \to \infty$, which implies the total number of shifted Fibonacci numbers is independent of the excess of positive to negative summands.*

Unfortunately, our arguments become involved and technical to handle all of the cases in Theorems 1.3 and 1.4. In order to highlight the ideas without getting bogged down in computations, in this note we concentrate on the most important special case of Theorem 1.3, namely the shifted Fibonacci numbers, and content ourselves with providing a sketch of some of the arguments in Theorem 1.4. We first describe our combinatorial perspective, which yields

$$p_n(k) = \frac{\binom{n-1-k}{k}}{\mathcal{F}_{n-1}}, \tag{1.5}$$

with $p_n(k)$ the probability that an $N \in [\mathcal{F}_n, \mathcal{F}_{n+1})$ has exactly $k+1$ summands in its Zeckendorf decomposition. All of our theorems follow from knowing this density. The difficulties in the general case are due to the fact that the corresponding formulas for the densities are far more involved; here we can easily determine the behavior by applying Stirling's formula.

We first use our explicit formula for $p_n(k)$ to prove Zeckendorf's Theorem, and then sketch how we can use it to prove Lekkerkerker's Theorem for the mean μ_n, as well as computing the variance σ_n^2. We then show that as $n \to \infty$, $p_n(k)$ converges to a Gaussian with mean μ_n and variance σ_n^2. While technically we could just immediately jump to the limiting behavior of the density, it would be very unmotivated not knowing the mean and the variance and either

choosing the correct values by divine inspiration, or having them fall out of the resulting algebra.

2. Combinatorial Perspective and Zeckendorf's Theorem

The key input in our analysis is counting the number of solutions to a well-known Diophantine equation. This problem is also known as the stars and bars problem, the cookie problem, or the simplest case of Waring's problem (see [12, 14]).

Lemma 2.1.

(1) *The number of ways of dividing n identical objects among p distinct people is $\binom{n+p-1}{p-1}$. Equivalently, this is the number of solutions to $y_1 + \cdots + y_p = n$ with each y_i a non-negative integer.*

(2) *More generally, the number of solutions to $y_1 + \cdots + y_p = n$ with $y_i \geq c_i$ (each c_i a non-negative integer) is $\binom{n-(c_1+\cdots+c_p)+p-1}{p-1}$.*

Proof. For (1): The two formulations are clearly equivalent; simply interpret y_i as the number of objects person i receives. To prove the claimed formula, imagine the m objects are in a row and we add $p-1$ items at the end. We now have $n+p-1$ objects. There is a one-to-one correspondence between assigning the n objects to the p people and choosing $p-1$ of $n+p-1$ objects. There are $\binom{n+p-1}{p-1}$ ways to choose $p-1$ of the $n+p-1$ items. All the items up to the first one chosen go to person 1, then the items up to the second one chosen go to person 2, and so on.

For (2): We may write $y_i = x_i + c_i$ with each $x_i \geq 0$ a non-negative integer. Our problem is equivalent to
$$(x_1 + c_1) + \cdots + (x_p + c_p) \ = \ n,$$
which becomes
$$x_1 + \cdots + x_p \ = \ n - (c_1 + \cdots + c_p), \quad x_i \geq 0,$$
whose solution is given by part (1). $\qquad\square$

There are two parts to Zeckendorf's Theorem: not only does a decomposition exist of any positive integer as a sum of non-consecutive Fibonacci numbers, but such a decomposition is unique. As our combinatorial approach does require this uniqueness as an input, we provide the standard proof below (or see, among others, [5, 16]).

Lemma 2.2 (Zeckendorf's Theorem - Uniqueness of Decomposition)**.** *If two sums of non-consecutive Fibonacci numbers are equal, then the two sums have the same summands.*

Proof. Assume
$$\mathcal{F}_{n_1} + \mathcal{F}_{n_2} + \cdots + \mathcal{F}_{n_k} = \mathcal{F}_{m_1} + \mathcal{F}_{m_2} + \cdots + \mathcal{F}_{m_\ell} \tag{2.1}$$
where $n_1 \geq n_2 \geq \cdots$ and $m_1 \geq m_2 \geq \cdots$. Without loss of generality we may assume $\mathcal{F}_{n_1} > \mathcal{F}_{m_1}$ (as otherwise we would just remove some summands). As each decomposition is of non-adjacent summands, if we add 1 to the decomposition on the right of (2.1), the largest it can be is $\mathcal{F}_{m_1} + \mathcal{F}_{m_2+1}$, which itself is at most $\mathcal{F}_{m_1+1}$. For example, $\mathcal{F}_6 + \mathcal{F}_4 + \mathcal{F}_2 + 1 = \mathcal{F}_6 + \mathcal{F}_4 + \mathcal{F}_3 = \mathcal{F}_6 + \mathcal{F}_5 = \mathcal{F}_7$. As $\mathcal{F}_{n_1} > \mathcal{F}_{m_1}$, we see that adding 1 to the right hand side of (2.1) yields a number at most $\mathcal{F}_{n_1}$; thus
$$\mathcal{F}_{n_1} + \mathcal{F}_{n_2} + \cdots + \mathcal{F}_{n_k} \ > \ \mathcal{F}_{m_1} + \mathcal{F}_{m_2} + \cdots + \mathcal{F}_{m_\ell}, \tag{2.2}$$
contradiction. $\qquad\square$

We are now in a position to prove the first part of Zeckendorf's Theorem. In the course of our proof we will derive (1.5), the claimed formula for $p_n(k)$.

Theorem 2.3 (Zeckendorf's Theorem - Existence of Decomposition). *Any natural number can be expressed as a sum of non-consecutive shifted Fibonacci numbers. Further, the probability an $N \in [\mathcal{F}_n, \mathcal{F}_{n+1})$ has exactly $k + 1$ summands in its Zeckendorf decomposition is $\binom{n-1-k}{k}/\mathcal{F}_{n-1}$ (in other words, the density function of the random variable $\mathcal{K}_n$ is $p_n(k) = \binom{n-1-k}{k}/\mathcal{F}_{n-1}$).*

Proof. Consider all N in $[\mathcal{F}_n, \mathcal{F}_{n+1})$. The number of integers in this interval is $\mathcal{F}_{n+1} - \mathcal{F}_n = \mathcal{F}_{n-1}$. We claim that each of these $\mathcal{F}_{n-1}$ integers can be expressed as a sum of a certain subset of $\{\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n\}$ with the properties that no two consecutive Fibonacci numbers appear in the sum and that $\mathcal{F}_n$ is one of the summands. From the arguments in the proof of Lemma 2.2, it is clear that $\mathcal{F}_n$ must appear in this sum since otherwise the sum would be too small to be an element of $[\mathcal{F}_n, \mathcal{F}_{n+1})$.

We now translate the previous claim into the combinatorial formulation of Lemma 2.1. Suppose we wish to have k summands in addition to the summand $\mathcal{F}_n$ in our sum, with no two summands adjacent. Clearly $k \leq \lfloor \frac{n-1}{2} \rfloor$. Choosing a valid set of k summands, say $\mathcal{F}_{m_1}, \ldots, \mathcal{F}_{m_k}$, is equivalent to choosing k indices $m_1, m_2, \ldots, m_k$ from the set $\{1, 2, \ldots, n-1\}$, with the property that $m_i < m_{1+1} - 1$ and $n - 1$ is not chosen (as otherwise we would have the adjacent summands $\mathcal{F}_{n-1}$ and $\mathcal{F}_n$).

We may assume $k > 0$, as there is only one way to choose no additional summands. We define the auxiliary sequence y_j as follows: $y_0 = m_1 - 1$, and for $1 \leq j \leq k$, $y_j = m_{j+1} - m_j - 1$ (as noted earlier, $m_{k+1} = n$). For example, with $n = 9$, $k = 3$, and the sequence $\mathcal{F}_1 + \mathcal{F}_3 + \mathcal{F}_6 + \mathcal{F}_9$, we have $y_0 = 0$, $y_1 = 1$, $y_2 = 2$ and $y_3 = 2$. Note y_0 is the number of indices before the first index chosen, and for $j > 0$, y_j equals the number of unused indices between m_j and m_{j+1}. Clearly we have $y_0 \geq 0$ and $y_j \geq 1$ for $j > 0$. Thus we have used $k + 1$ indices (including the required index n associated with the largest summand, $\mathcal{F}_n$) and $y_0 + y_1 + \cdots + y_k$ unused indices from $\{1, 2, \ldots, n\}$. Hence, we have

$$(k + 1) + y_0 + y_1 + \cdots + y_k \;=\; n. \tag{2.3}$$

If we make the change of variables $x_0 = y_0$ and $x_j = y_j - 1$ for $j > 0$, then we have $x_j \geq 0$ for all j with no other constraints on these variables other than they must satisfy the identity implicitly given by (2.3); that is

$$x_0 + \cdots + x_k \;=\; n - 1 - 2k. \tag{2.4}$$

In other words, in view of Lemma 2.2 we have a bijection between the set of Zeckendorf decompositions with $k + 1$ summands having $\mathcal{F}_n$ as its largest summand and the set of all non-negative integer solutions to (2.4). By Lemma 2.1, the number of solutions to (2.4) is $\binom{n-1-2k+(k+1-1)}{k+1-1} = \binom{n-1-k}{k}$. Thus the number of Zeckendorf decompositions having largest summand $\mathcal{F}_n$ is precisely

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k}, \tag{2.5}$$

which, by a well-known identity for binomial sums, equals $\mathcal{F}_{n-1}$ (for a proof, see either [9] or Appendix A of [8]). As remarked, by Lemma 2.2, each one of these sequences gives rise to a distinct Zeckendorf sum. Thus the number of Zeckendorf decompositions in the interval $[\mathcal{F}_n, \mathcal{F}_{n+1})$ is equal to $\mathcal{F}_{n-1}$, which is the total number of integers in that interval. As n was arbitrary, and these intervals partition the set of natural numbers, we have shown that every

natural number has a unique Zeckendorf decomposition, and the number of N in $[\mathcal{F}_n, \mathcal{F}_{n+1})$ with exactly $k+1$ summands in its decomposition is $\binom{n-1-k}{k}$. In other words, the probability of a number in this interval having precisely $k+1$ summands is $\binom{n-1-k}{k}/\mathcal{F}_{n-1}$. $\qquad\square$

3. Lekkerkerker's Theorem and the Variance

We sketch how our approach easily yields Lekkerkerker's Theorem. We only provide a sketch as, of course, Lekkerkerker's Theorem follows immediately from our proof of the Gaussian behavior. We highlight the key steps as Lekkerkerker's Theorem is of interest in its own right, and it is nice to have a new, elementary proof of it (our proof of the Gaussian behavior will involve Stirling's formula, and is different than the arguments below).

The average number of summands needed in the Zeckendorf decomposition is just

$$
\begin{aligned}
\mu_n &= \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} (k+1) \frac{\binom{n-1-k}{k}}{\mathcal{F}_{n-1}} \\
&= 1 + \frac{1}{\mathcal{F}_{n-1}} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k} \\
&= 1 + \frac{\mathcal{E}(n)}{\mathcal{F}_{n-1}}.
\end{aligned}
\tag{3.1}
$$

Thus the problem is reduced to computing

$$
\mathcal{E}(n) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} k \binom{n-1-k}{k}.
\tag{3.2}
$$

We can determine a closed-form expression for $\mathcal{E}(n)$ by first showing that it satisfies a certain recurrence relation.

Lemma 3.1 (Recurrence relation for $\mathcal{E}(n)$)**.** *We have*

$$
\mathcal{E}(n) + \mathcal{E}(n-2) = (n-2)\mathcal{F}_{n-3}.
\tag{3.3}
$$

The proof follows from straightforward algebra (see Appendix A of [8]). Solving the recurrence relation yields the following.

Lemma 3.2 (Formula for $\mathcal{E}(n)$)**.** *We have*

$$
\mathcal{E}(n) = \frac{n\mathcal{F}_{n-1}}{\alpha^2 + 1} + O(\mathcal{F}_{n-2}).
\tag{3.4}
$$

The proof follows from using telescoping sums to get an expression for $\mathcal{E}(n)$, which is then evaluated by applying Binet's formula and differentiating identities (see Appendix A of [8] for the detailed computations).

Lekkerkerker's Theorem now follows immediately by substituting the result for $\mathcal{E}(n)$ from Lemma 3.2 into the equation for the mean, (3.1). We have the following result.

Theorem 3.3. *The average number of non-consecutive shifted Fibonacci summands used in representing integers in $[\mathcal{F}_n, \mathcal{F}_{n+1})$ is*

$$
\mathbb{E}[K_n] = \mu_n = \frac{5 - \sqrt{5}}{10}\, n - \frac{2}{5} = \frac{1}{\alpha^2 + 1}\, n - \frac{2}{5} = \mathbb{E}[\mathcal{K}_n] + 1.
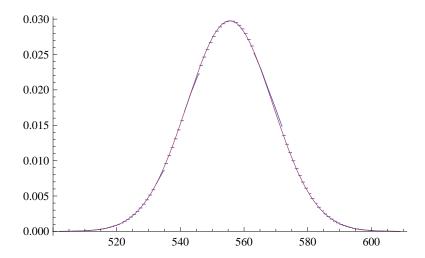\tag{3.5}
$$

FIGURE 1. Plot of the probability $p_{2010}(k)$ of having exactly $k+1$ summands in the Zeckendorf decompositions of integers in $[\mathcal{F}_{2010}, \mathcal{F}_{2011})$ (with $p_n(k) = \binom{n-1-k}{k}/\mathcal{F}_{n-1}$) against the Gaussian with mean $\mu_{2010} \approx 555.55$ and variance $\sigma_{2010}^2 \approx 179.78$.

A similar calculation shows the following.

Theorem 3.4. *The variance in the number of non-consecutive shifted Fibonacci numbers used in representing integers in $[\mathcal{F}_n, \mathcal{F}_{n+1})$ is*

$$\mathrm{Var}(K_n) = \sigma_n^2 = \frac{1}{5\sqrt{5}}\ n - \frac{2}{25} = \frac{\alpha}{5(\alpha+2)}\ n - \frac{2}{25} = \mathrm{Var}(\mathcal{K}_n). \tag{3.6}$$

4. GAUSSIAN BEHAVIOR

In the last section we computed the mean by evaluating certain combinatorial sums. We could similarly derive an explicit formula for the variance, or more generally, any moment. We choose instead to analyze the density $p_n(k)$ in greater detail, and show that it converges pointwise to a Gaussian with mean μ_n approximately $\frac{n}{\alpha^2+1}$ and variance $\sigma_n^2 \approx \frac{\alpha n}{5(\alpha+2)}$. In the course of proving this convergence, the mean and the standard deviation naturally fall out of the calculation. While this does make the previous section superfluous, we chose to include it as it provides an elementary proof of Lekkerkerker, as well as telling us what the mean and variance are, which are a great aid in performing the Stirling analysis below. If we didn't know μ_n and σ_n we would just keep these as initially free parameters, and then choose the values appropriately to ensure the limits below exist.

Before delving into the proof of Theorem 1.3, we provide some evidence by looking at the number of summands in the Zeckendorf decomposition for integers in $[\mathcal{F}_{2010}, \mathcal{F}_{2011})$ (see Figure 1); the fit is visually striking.

4.1. **Preliminaries.** The computation is a little cleaner if instead of studying the interval $[\mathcal{F}_n, \mathcal{F}_{n+1})$ we instead study $[\mathcal{F}_{n+1}, \mathcal{F}_{n+2})$. The density is now

$$p_{n+1}(k) = \frac{\binom{n-k}{k}}{\mathcal{F}_n}. \tag{4.1}$$

We list some useful expansions:

$$\mu_{n+1} = \frac{n}{\alpha + 2} + O(1)$$

$$\sigma_{n+1}^2 = \frac{\alpha n}{5(\alpha + 2)} + O(1)$$

$$\mathcal{F}_n = \frac{\alpha}{\sqrt{5}}\alpha^n + O\left((1 - \alpha)^n\right). \tag{4.2}$$

We expand $\binom{n-k}{k}$ using Stirling's formula (see [15] for a proof), which says for large m that

$$m! = m^m e^{-m}\sqrt{2\pi m}\left(1 + \frac{1}{12m} + \frac{1}{288m^2} - \frac{139}{51840m^3} + \cdots\right)$$

$$= m^m e^{-m}\sqrt{2\pi m}\left(1 + O\left(\frac{1}{m}\right)\right). \tag{4.3}$$

While we have not proved the variance is of size n, a similar calculation as that for the mean yields it without any trouble; in the interest of space we merely state the result in Theorem 3.3. We could also get the right order of magnitude for the variance by the Stirling computation that follows. As the mean and variance are both of size n, for any fixed $\epsilon > 0$ as $n \to \infty$ the set of k with $|k - \mu_{n+1}| > n^{1/2+\epsilon}$ has negligible probability. This follows by Chebyshev's inequality. As we are on the order of n^ϵ standard deviations from the mean, the probability is at most $O(1/n^{2\epsilon})$, which tends to zero with n. Thus we need only worry about analyzing $p_{n+1}(k)$ for $|k - \mu_{n+1}| \le n^{1/2+\epsilon}$. For such k, we have k, $n - k$ and $n - 2k$ are all of size n and hence large, and consequently the first term in Stirling's formula suffices for our analysis.

After some simple algebra, which includes using Binet's formula (see (1.1)) for $\mathcal{F}_n$, we find

$$p_{n+1}(k) = \binom{n-k}{k}\frac{1}{\mathcal{F}_n}$$

$$= \frac{(n-k)!}{k!(n-2k)!}\frac{1}{\mathcal{F}_n}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{(n-k)^{n-k+\frac{1}{2}}\left(1 + O(\frac{1}{n-k})\right)}{k^{k+\frac{1}{2}}\left(1 + O\left(\frac{1}{k}\right)\right)(n-2k)^{n-2k+\frac{1}{2}}\left(1 + O(\frac{1}{n-2k})\right)}\frac{\sqrt{5}}{\alpha \cdot \alpha^n\left(1 + O\left(\frac{1}{\alpha^n}\right)\right)}$$

$$= \frac{1}{\sqrt{2\pi}}\frac{(n-k)^{n-k+\frac{1}{2}}}{k^{k+\frac{1}{2}}(n-2k)^{n-2k+\frac{1}{2}}}\frac{\sqrt{5}}{\alpha \cdot \alpha^n}\left(1 + O\left(\frac{1}{n}\right)\right)$$

$$= f_{n+1}(k)\left(1 + O\left(\frac{1}{n}\right)\right). \tag{4.4}$$

It suffices to analyze $f_{n+1}(k)$ for $|k - \mu_{n+1}| \le n^{1/2+\epsilon}$, as the $O(1/n)$ term leads to negligible changes in the density function. We now split off the terms that exponentially depend on n, and write

$$f_{n+1}(k) = \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\alpha}\right)\left(\alpha^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}\right)$$

$$= N_n(k)S_n(k). \tag{4.5}$$

We change variables and replace k with its distance from the mean in units of the standard deviation, σ_{n+1}. Thus we write

$$k = \mu_{n+1} + x\sigma_{n+1}, \tag{4.6}$$

and

$$f_{n+1}(k)dk = f_{n+1}(\mu_{n+1} + \sigma_{n+1}x)\sigma_{n+1}dx. \tag{4.7}$$

Note the change of variables in (4.6) results in dk transforming to $\sigma_{n+1}dx$. In these new variables the main action occurs at $x = 0$, and the scale is on the order of 1; in other words, once x is large (such as n^ϵ) then we are many standard deviations away and the density is negligible.

The next few pages present the detailed computation. While the computations are long in places, the basic idea (the combinatorial perspective) is straightforward: viewing the problem combinatorially yields an explicit density function, whose large n asymptotics follow from Stirling's formula and careful bookkeeping.

4.2. Analysis of $N_n(k)$.

Lemma 4.1. *For any $\epsilon > 0$ we have*

$$N_n(k) = \frac{1}{\sqrt{2\pi\sigma_{n+1}^2}} \cdot \left(1 + O\left(n^{-1/2+\epsilon}\right)\right). \tag{4.8}$$

Let $C = \frac{1}{\alpha+2}$, so $\mu_{n+1} = Cn + O(1)$, and recall $\sigma_{n+1}^2 = \frac{\alpha n}{5(\alpha+2)} + O(1)$. We use a change of variable and some algebra to simplify $N_n(k)$ (defined in (4.5)). We set

$$u = \frac{\sigma_{n+1}}{n}x \approx \frac{x}{\sqrt{n}} \ll O(n^{-1/2+\epsilon}), \tag{4.9}$$

and find

$$
\begin{aligned}
N_n(k) &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n-k}{k(n-2k)}}\frac{\sqrt{5}}{\alpha} \\
&= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-k/n}{(k/n)(1-2k/n)}}\frac{\sqrt{5}}{\alpha} \\
&= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-(\mu_{n+1}+\sigma_{n+1}x)/n}{((\mu_{n+1}+\sigma_{n+1}x)/n)(1-2(\mu_{n+1}+\sigma_{n+1}x)/n)}}\frac{\sqrt{5}}{\alpha} \\
&= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C-u}{(C+u)(1-2C-2u)}}\frac{\sqrt{5}}{\alpha}\cdot\left(1+\frac{1}{n}\right), \tag{4.10}
\end{aligned}
$$

where the last error arises from replacing μ_{n+1} with $Cn + O(1)$. In fact, as $u = O(n^{-1/2+\epsilon})$ we may drop the u's at the cost of replacing the error term $O(n^{-1})$ with $O(n^{-1/2+\epsilon})$. The following relations help simplify our expression for $N_n(k)$:

$$\alpha + 1 = \alpha^2, \quad C = \frac{1}{\alpha+2}, \quad 1 - C = \frac{\alpha+1}{\alpha+2}, \quad 1 - 2C = \frac{\alpha}{\alpha+2}.$$

Using these, as well as $\sigma_{n+1}^2 = \frac{\alpha n}{5(\alpha+2)} + O(1)$, we find

$$
\begin{aligned}
N_n(k) &\approx \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{1-C}{C(1-2C)}} \frac{\sqrt{5}}{\alpha} \cdot \left(1 + O\left(n^{-1/2+\epsilon}\right)\right) \\
&= \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{(\alpha+1)(\alpha+2)}{\alpha}} \frac{\sqrt{5}}{\alpha} \cdot \left(1 + O\left(n^{-1/2+\epsilon}\right)\right) \\
&= \frac{1}{\sqrt{2\pi n}} \sqrt{\frac{5(\alpha+2)}{\alpha}} \cdot \left(1 + O\left(n^{-1/2+\epsilon}\right)\right) \\
&= \frac{1}{\sqrt{2\pi \sigma_{n+1}^2}} \cdot \left(1 + O\left(n^{-1/2+\epsilon}\right)\right),
\end{aligned}
\tag{4.11}
$$

which is recognizable as the normalization constant for a Gaussian with variance σ_{n+1}^2.

4.3. Analysis of $S_n(k)$.

Lemma 4.2. *For any $\epsilon > 0$ we have $S_n(k) = \exp(-x^2/2) \exp\left(O(n^{-1/2+3\epsilon})\right)$.*

Proof. To determine the size of $S_n(k)$ (defined in (4.5)), we take logarithms and once again change variables by $k = \mu_{n+1} + x\sigma_{n+1}$. We find

$$
\begin{aligned}
\log S_n(k) &= \log\left(\alpha^{-n} \frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}\right) \\
&= -n\log\alpha + (n-k)\log(n-k) - k\log k - (n-2k)\log(n-2k) \\
&= -n\log\alpha + (n - (\mu_{n+1} + x\sigma_{n+1}))\log(n - (\mu_{n+1} + x\sigma_{n+1})) \\
&\quad - (\mu_{n+1} + x\sigma_{n+1})\log(\mu_{n+1} + x\sigma_{n+1}) \\
&\quad - (n - 2(\mu_{n+1} + x\sigma_{n+1}))\log(n - 2(\mu_{n+1} + x\sigma_{n+1})) \\
&= -n\log\alpha \\
&\quad + (n - (\mu_{n+1} + x\sigma_{n+1}))\left(\log(n-\mu_{n+1}) + \log\left(1 - \frac{x\sigma_{n+1}}{n-\mu_{n+1}}\right)\right) \\
&\quad - (\mu_{n+1} + x\sigma_{n+1})\left(\log(\mu_{n+1}) + \log\left(1 + \frac{x\sigma_{n+1}}{\mu_{n+1}}\right)\right) \\
&\quad - (n - 2(\mu_{n+1} + x\sigma_{n+1}))\left(\log(n-2\mu_{n+1}) + \log\left(1 - \frac{2x\sigma_{n+1}}{n-2\mu_{n+1}}\right)\right) \\
&= -n\log\alpha \\
&\quad + (n - (\mu_{n+1} + x\sigma_{n+1}))\left(\log\left(\frac{n}{\mu_{n+1}} - 1\right) + \log\left(1 - \frac{x\sigma_{n+1}}{n-\mu_{n+1}}\right)\right) \\
&\quad - (\mu_{n+1} + x\sigma_{n+1})\log\left(1 + \frac{x\sigma_{n+1}}{\mu_{n+1}}\right) \\
&\quad - (n - 2(\mu_{n+1} + x\sigma_{n+1}))\left(\log\left(\frac{n}{\mu_{n+1}} - 2\right) + \log\left(1 - \frac{2x\sigma_{n+1}}{n-2\mu_{n+1}}\right)\right). \tag{4.12}
\end{aligned}
$$

We may simplify the expression above by replacing any μ_{n+1} inside a logarithm with Cn. This is because the resulting Taylor expansion of the logarithms will yield a term of size $1/n^2$. The largest this can be multiplied by is n, which leads to an error at most $O(1/n)$. We

exponentiate to get $S_n(k)$, which leads to an error factor of size $1 - \exp(O(1/n))$, which as $n \to \infty$ is just of size $1/n$. We have

$$\log\left(\frac{n}{\mu_{n+1}} - 1\right) = \log\frac{n - \mu_{n+1}}{\mu_{n+1}} = \log\frac{1 - C}{C} + O\left(\frac{1}{n^2}\right) = 2\log\alpha + O\left(\frac{1}{n^2}\right),$$

as $(1 - C)/C = \alpha + 1 = \alpha^2$. Similarly

$$\log\left(\frac{n}{\mu_{n+1}} - 2\right) = \log\frac{n - 2\mu_{n+1}}{\mu_{n+1}} = \log\frac{1 - 2C}{C} + O\left(\frac{1}{n^2}\right) = \log\alpha + O\left(\frac{1}{n^2}\right).$$

Substituting these into (4.12) yields

$$\begin{aligned}
\log S_n(k) &+ O\left(\frac{1}{n}\right) \\
&= -n\log\alpha + (n - (\mu_{n+1} + x\sigma_{n+1}))\left(2\log\alpha + \log\left(1 - \frac{x\sigma_{n+1}}{(1 - C)n}\right)\right) \\
&\quad - (\mu_{n+1} + x\sigma_{n+1})\log\left(1 + \frac{x\sigma_{n+1}}{Cn}\right) \\
&\quad - (n - 2(\mu_{n+1} + x\sigma_{n+1}))\left(\log\alpha + \log\left(1 - \frac{2x\sigma_{n+1}}{(1 - 2C)n}\right)\right).
\end{aligned} \tag{4.13}$$

We note that the coefficient of the $\log\alpha$ term is zero, so

$$\begin{aligned}
\log S_n(k) + O\left(\frac{1}{n}\right) &= (n - (\mu_{n+1} + x\sigma_{n+1}))\log\left(1 - \frac{x\sigma_{n+1}}{(1 - C)n}\right) \\
&\quad - (\mu_{n+1} + x\sigma_{n+1})\log\left(1 + \frac{x\sigma_{n+1}}{Cn}\right) \\
&\quad - (n - 2(\mu_{n+1} + x\sigma_{n+1}))\log\left(1 - \frac{2x\sigma_{n+1}}{(1 - 2C)n}\right) \\
&= ((1 - C)n - x\sigma_{n+1}))\log\left(1 - \frac{x\sigma_{n+1}}{(1 - C)n}\right) \\
&\quad - (Cn + x\sigma_{n+1})\log\left(1 + \frac{x\sigma_{n+1}}{Cn}\right) \\
&\quad - ((1 - 2C)n - 2x\sigma_{n+1}))\log\left(1 - \frac{2x\sigma_{n+1}}{(1 - 2C)n}\right).
\end{aligned} \tag{4.14}$$

Let $u = x\sigma_{n+1}/n$. Note $u = O(n^{-1/2+\epsilon})$, and thus when we expand the logarithms above, we never need to keep more than the u^2 terms, as anything further will be small, even upon multiplication by n. Expanding gives

$$\begin{aligned}
\log S_n(k) &+ O\left(n^{-1/2+3\epsilon}\right) \\
&= (1 - C)n\left(-\frac{u}{1 - C} - \frac{u^2}{2(1 - C)^2}\right) - un\left(-\frac{u}{1 - C}\right) \\
&\quad - Cn\left(\frac{u}{C} - \frac{u^2}{2C^2}\right) - un\left(\frac{u}{C}\right) \\
&\quad - (1 - 2C)n\left(-\frac{2u}{1 - 2C} - \frac{4u^2}{2(1 - 2C)^2}\right) + 2un\left(-\frac{2u}{1 - 2C}\right).
\end{aligned} \tag{4.15}$$

Note all the un terms cancel, and all that survives are the u^2n terms. Thus

$$\log S_n(k) + O\left(n^{-1/2+3\epsilon}\right)$$
$$= \left[-\frac{1}{2(1-C)} + \frac{1}{1-C} + \frac{1}{2C} - \frac{1}{C} + \frac{2}{1-2C} - \frac{4}{1-2C}\right] u^2n$$
$$= -\frac{u^2n}{2C - 6C^2 + 4C^3}. \tag{4.16}$$

As $C = \frac{1}{\alpha+2}$ and $u = x\sigma_{n+1}/n$ with $\sigma_{n+1}^2 = \frac{\alpha n}{5(\alpha+2)} + O(1)$, simplifying the above yields

$$\log S_n(k) + O\left(n^{-1/2+3\epsilon}\right) = -\frac{x^2}{2}, \tag{4.17}$$

and thus exponentiating gives

$$S_n(k) = \exp(-x^2/2)\exp\left(O(n^{-1/2+3\epsilon})\right). \tag{4.18}$$

$\square$

4.4. **Proof of Theorem 1.3.** Using the results from the previous subsections (Lemmas 4.1 and 4.2, and the change of variable argument on why a factor of σ_{n+1} enters), we can now prove the convergence to a Gaussian.

Proof of Theorem 1.3. The only item left to prove in Theorem 1.3 is the convergence to the Gaussian. We have

$$\begin{aligned} p_{n+1}(k)dk &= p_{n+1}\left(\mu_{n+1} + \sigma_{n+1}x\right)\sigma_{n+1}dx \\ &= N_n(k)S_n(k)\sigma_{n+1}\left(1 + O(n^{-1/2+3\epsilon})\right)dx \\ &= \frac{1}{\sqrt{2\pi\sigma_{n+1}^2}} \cdot e^{-x^2/2} \cdot \sigma_{n+1}\left(1 + O(n^{-1/2+3\epsilon})\right)dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\left(1 + O(n^{-1/2+3\epsilon})\right)dx, \end{aligned} \tag{4.19}$$

which clearly converges to the standard normal as $n \to \infty$. $\square$

5. FAR-DIFFERENCE REPRESENTATIONS

We now consider the problem of the far-difference representations. We are studying the number of positive and negative shifted Fibonacci summands in the decomposition of integers in $[S_n, S_{n+1})$, where $S_n = \sum_{0<n-4i\le n} \mathcal{F}_{n-4i}$. Using combinatorial arguments as in Lemma 2.1 (which involve solving a variant of this Diophantine problem inside a variant of this Diophantine problem), we can come up with a formula for the joint density function $p_n(k,\ell)$ for the number of integers in $[S_n, S_{n+1})$ with exactly k positive Fibonacci summands and exactly ℓ

negative Fibonacci summands. After a lot of algebra, we obtain the formula:

$$
\begin{aligned}
p_n(k,\ell) = \sum_{r=0}^{k} \binom{k-1}{k-r} \Bigg[& \binom{\ell-1}{r-2}\binom{n-3(k+\ell)+2r-3}{k+\ell} \\
+ & \binom{\ell-1}{r-1}\binom{n-3(k+\ell)+2r-2}{k+\ell} + \binom{\ell-1}{r-1}\binom{n-3(k+\ell)+2r-1}{k+\ell} \\
+ & \binom{\ell-1}{r}\binom{n-3(k+\ell)+2r}{k+\ell} \Bigg].
\end{aligned}
\tag{5.1}
$$

If we could derive good asymptotics for this formula, we could calculate the limiting density directly, but this does not seem feasible. With enough patience, this formula can be used to calculate any particular joint moment, but this approach quickly becomes impractical due to the combinatorial complexity. Unlike our expression for the standard Fibonacci case, here we have sums and products of binomial coefficients that would need to be simplified before we can fruitfully apply Stirling's formula. As the generating function technique is able to handle this problem, we invite the reader to see [13] for a complete analysis of this problem.

6. Conclusion and Future Research

Our combinatorial viewpoint has allowed us to extend previous work and obtain Gaussian behavior for the number of summands for a large class of recurrence relations. This is just the first of many questions one can ask. Others, which we hope to return to at a later date, include:

(1) Lekkerkerker's Theorem, and the Gaussian extension, are for the behavior in intervals $[\mathcal{F}_n, \mathcal{F}_{n+1})$. Do the limits exist if we consider other intervals, say $[\mathcal{F}_n + g_1(\mathcal{F}_n), \mathcal{F}_n + g_2(\mathcal{F}_n))$ for some functions g_1 and g_2? If yes, what must be true about the growth rates of g_1 and g_2?

(2) For the generalized recurrence relations, what happens if instead of looking at $\sum_{i=1}^{n} a_i$ we study $\sum_{i=1}^{n} \min(1, a_i)$? In other words, we only care about how many distinct H_i's occur in the decomposition.

(3) What can we say about the distribution of the largest gap between summands in the Zeckendorf decomposition? Appropriately normalized, how does the distribution of gaps between the summands behave?

7. Acknowledgement

References

[1] H. Alpert, *Differences of multiple Fibonacci numbers*, INTEGERS: **9** (2009), 745–749.

[2] E. Burger, personal communication, 2010.

[3] D. E. Daykin, *Representation of natural numbers as sums of generalized Fibonacci numbers*, J. London Mathematical Society, **35** (1960), 143–160.

[4] P. Erdős and M. Kac, *The Gaussian law of errors in the theory of additive number theoretic functions*, American Journal of Mathematics, **62.1/4** (1940), 738–742.

[5] A. S. Fraenkel, *Systems of numeration*, Amer. Math. Monthly, **92.2** (1985), 105–114.

[6] V. E. Hoggatt, *Generalized Zeckendorf theorem*, The Fibonacci Quarterly, **10.1** (1972), (special issue on representations), 89–94.

[7] T. J. Keller, *Generalizations of Zeckendorfs theorem*, The Fibonacci Quarterly, **10.1** (1972), (special issue on representations), 95–102.

[8] G. Kopp, M. Koloğlu, S. J. Miller, and Y. Wang, *On the number of summands in Zeckendorf decompositions*, `http://arxiv.org/abs/1008.3204.`

[9] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley-Interscience, New York, 2001.

[10] C. G. Lekkerkerker, *Voorstelling van natuurlykxe getallen door een som van getallen van Fibonacci*, Simon Stevin, **29** (1951–1952), 190–195.

[11] T. Lengyel, *A counting based proof of the generalized Zeckendorf's theorem*, The Fibonacci Quarterly, **44.4** (2006), 324–325.

[12] S. J. Miller and R. Takloo-Bighash, *An Invitation to Modern Number Theory*, Princeton University Press, Princeton, NJ, 2006.

[13] S. J. Miller and Y. Wang, *From Fibonacci numbers to central limit type theorems*, submitted, `http://arxiv.org/abs/1008.3202.`

[14] M. Nathanson, *Additive Number Theory: The Classical Bases*, Graduate Texts in Mathematics, Springer-Verlag, New York, 1996.

[15] E. Whittaker and G. Watson, *A Course of Modern Analysis*, 4th edition, Cambridge University Press, Cambridge, 1996.

[16] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Roy. Sci. Liége, **41** (1972), 179–182.

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