INTEGER SEQUENCES GENERATED BY $x_{n+1} = \frac{x_n^2+A}{x_{n-1}}$

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Abstract. We describe certain elementary sequences which are integer valued and characterize the integral sequences for the special example $x_{n+1}x_{n-1} = x_n^2 + 1$; this is related to the alternate terms of the Fibonacci sequence.

1. Introduction

We consider the sequences generated by the non-linear equation for $n \geq 1$

$$x_{n+2}x_n = x_{n+1}^2 + A$$

with constant $A \neq 0$ and initial values x_1, x_2 specified. First, we want to know for a given A, which integer values of x_1 and x_2 will give a sequence consisting only of integers. We call the sequence integral if this happens. It is known that the sequences generated by this equation will have denominators of the form $x_1^{n-1}x_2^{n-2}$ in general, that is as a formal sequence (this is the Laurent phenomenon as discussed in [1]). However under special circumstances the sequence will be integral.

Secondly, we consider the uniqueness (up to sign and shift) of the integral sequences for a fixed value of A. Can one characterize the values of A for which these integral sequences are unique? In this regard we prove that when $A = 1$ the sequence is essentially unique and is just a signed variation on the alternate terms of the Fibonacci sequence (Corollary 4.2). It would be interesting to know if there are infinitely many A for which integral sequences are essentially unique.

However, there are infinitely many cases when uniqueness fails; for example let $A = -k^2 + k^3 - 1$, k an integer, then the sequences with $x_1 = 1$, $x_2 = 1$ or with $x_1 = 1$, $x_2 = k$ are distinct (see 2.1).

2. A-Sequence

We denote by $\Sigma = \Sigma_A(x_1, x_2)$ the sequence determined by $x_{n+1}x_{n-1} = x_n^2 + A$; we refer to this as an A-sequence. We first show that the sequences are linearly recursive.

Proposition 2.1. *Suppose that x_n is an A-sequence and let $\mu = \frac{x_2^2+x_1^2+A}{x_1x_2}$. Then the sequence satisfies $x_{n+1} = \mu x_n - x_{n-1}$.*

Proof. We show that $\frac{x_{n+1}+x_{n-1}}{x_n}$ is constant and equal to μ by induction. Certainly this equality is valid for $n = 2$: $\frac{x_3+x_1}{x_2} = \mu$. Now assume it is valid for n. Then we have

$$\frac{x_{n+2}+x_n}{x_{n+1}} = \frac{x_{n+1}^2 + x_n^2 + A}{x_{n+1}x_n}$$

because x_n is an A-sequence. Then,

$$\begin{aligned}\mu - \frac{x_{n+2}+x_n}{x_{n+1}} &= \frac{x_{n+1}+x_{n-1}}{x_n} - \frac{x_{n+1}^2+x_n^2+A}{x_{n+1}x_n} \\ &= \frac{x_{n+1}^2+x_{n-1}x_{n+1}-x_{n+1}^2-x_n^2-A}{x_{n+1}x_n} \\ &= \frac{x_{n-1}x_{n+1}-x_n^2-A}{x_{n+1}x_n} \\ &= 0.\end{aligned}$$

□

2.1. **Integral Examples.** Here is a method to generate integral sequences. Let x_1, x_2 determine μ as before, say $x_1, x_2 \in \{r, s\} \subset \mathbb{Z}$ with $\frac{s+1}{r} \in \mathbb{Z}$, and $A = s - r^2$ then $\mu = \frac{r^2+s^2+s-r^2}{rs} = \frac{s+1}{r}$. Certainly r and s has no common factor and the sequence $\Sigma_A(r, s)$ consists of integers. For example with $x_1 = r = 1$, $x_2 = s$, $\mu = s + 1$, $A = s - 1$, we obtain an integral sequence for any integer value of A.

2.2. **Other Sequences.** The sequences $x_{n+1}x_{n-1} = x_n^2 + Bx_n + A$, are linearly recursive of degree 3 with characteristic equation $X^3 - \mu X^2 + \mu X - 1$ with $\mu = \frac{x_1^2+x_2^2+x_1x_2+B(x_1+x_2)+A}{x_1x_2}$ when $B \neq 0$.

The sequences $x_{n+1}x_{n-2} = x_nx_{n+1} + A$ satisfy the linear recurrence $x_{n+1} = \mu x_{n-1} + x_{n-3}$ and $\mu = \frac{x_1(x_0^2+x_2^2)+A(x_0+x_2)}{x_0x_1x_2}$.

In a very similar way one can show that these sequences are linear. These also satisfy the Laurent conditions of [1]. We leave the details to the interested reader.

3. Pell's Equation and Integral A-Sequences

Suppose that $x_1 = a, \mu \in \mathbb{Z}$, then using the formula for μ we have an integer equation

$$x^2 - a\mu x + A + a^2 = 0$$

which will have an integer solution $x = x_2 \in \{\frac{a\mu}{2} \pm \frac{\sqrt{(a\mu)^2-4A-4a^2}}{2}\}$ if and only if the discriminant is an integer square c^2 and $a\mu \pm c$ is even.

Hence, we also have integer solutions $X = c$, $Y = a$ to Pell's equation

$$X^2 - (\mu^2 - 4)Y^2 = -4A. \tag{3.1}$$

Proposition 3.1. *The A-sequence is integral if and only if there are integer solutions when c is even to: $X^2 - \frac{\mu^2-4}{4}Y^2 = -A$ when μ is even or $X^2 - (\mu^2-4)Y^2 = -A$ when μ is odd; or when c is odd then μ is odd and $X^2 - (\mu^2-4)Y^2 = -4A$ has a solution with X odd.*

Proof. With the notation as above, suppose first that c is even. If $\mu^2 - 4$ is odd then a is even so the equation reduces to $X^2 - (\mu^2-4)Y^2 = -A$. If $\mu^2 - 4$ is even then μ is also even and then the equation reduces to $X^2 - ((\frac{\mu}{2})^2 - 1)Y^2 = -A$.

If, however, c is odd then $\mu^2 - 4$ and a are both odd and the equation remains as $X^2 - (\mu^2-4)Y^2 = -4A$.

Conversely if we have solutions to Pell's equation 3.1 above then we can make an A-sequence integral solution using the solution for $x_1 = Y$ or $x_1 = 2Y$ and then solve for x_2 using the quadratic formula given $\mu^2 - 4$ with known x_1, A. The equation is simply

$$x_2^2 + A + a^2 - \mu a x_2 = 0, \tag{3.2}$$

and thus we have proven the proposition. □

4. Uniqueness Property for $A = 1$

There may not be a unit of norm -1 or -4 in the associated ring for Pell's equation, $X^2 - rY^2 = -1$, $X^2 - rY^2 = -4$. The existence of the unit of norm -1 depends on whether or not the period of the continued fraction of $\sqrt{r}$ is odd [2].

If $r = \mu$ is odd and $r > 3$ then $\sqrt{r^2 - 4}$ has even period since

$$\sqrt{r^2 - 4} = \left(r - 1; \overline{1, \frac{r-3}{2}, 2, \frac{r-3}{2}, 1, 2r - 2}\right).$$

If $s = \frac{\mu}{2}$ is an integer then for $s \geq 2$, $\sqrt{s^2 - 1} = (s - 1; \overline{1, 2s - 2})$ has even period.

Theorem 4.1. *If $A = 1$ then the integral A-sequences exist if and only if $\mu = \pm 3$. Any integer solution to $X^2 - 5Y^2 = -1$ gives an integral A-sequence with $x_1 = Y$ and x_2 a solution to the quadratic equation $x_2^2 - \mu x_1 x_2 + 1 + x_1^2 = 0$.*

Proof. We have shown above there are no solutions to Pell's equation $X^2 - (\mu^2 - 4)Y^2 = -1$, $\mu \neq \pm 3$. Also we have shown above there are no solutions to $X^2 - \frac{\mu^2 - 4}{4} Y^2 = -1$ for μ even and $\frac{\mu}{2} \geq 2$. For the last case we consider solutions to Pell's equation

$$X^2 - (\mu^2 - 4)Y^2 = -4$$

with $X = c$ odd; hence μ is odd and $Y = a$ is also odd. We may assume that $\mu^2 - 4$ is square-free since any square factor can be absorbed into the solution for Y. In this situation using the congruence $\pmod 4$ we see that the Pell's equation has no solution if $\mu^2 - 4 \equiv 3 \pmod 4$.

Suppose then that $D = \mu^2 - 4 \equiv 1 \pmod 4$. The algebraic integers $\mathbb{Z}_D$ in the field $\mathbb{Q}(\sqrt{D})$ properly contains the ring $\mathbb{Z}[\sqrt{D}]$. If the fundamental unit of $\mathbb{Z}_D$ does not lie in $\mathbb{Z}[\sqrt{D}]$ then we get the desired solution to Pell's equation. Conversely, if we have the desired solution X, Y odd then we get a unit in $\mathbb{Z}_D$ which does not lie in $\mathbb{Z}[\sqrt{D}]$. However the cube of this unit lies in the ring $\mathbb{Z}[\sqrt{D}]$ which means that there is a solution to Pell's equation $x^2 - (\mu^2 - 4)y^2 = -1$; but this is impossible since the period is even. (Note that $\mu^2 - 4 \equiv 1 \pmod 8$ is impossible since there is no solution to $x^2 \equiv 5 \pmod 8$. Also $\mu^2 - 4 \equiv 5 \pmod 8$ is used to show that the cube of a unit in the larger ring lies in the smaller ring.) □

If we also reverse the sequence to include x_n, $n \leq 0$ then essentially there are just 4 sequences when $A = 1$, ignoring the exact starting place.

The solutions for $r = \mu = \pm 3$ correspond to *odd* powers of the fundamental unit $\alpha = \frac{1+\sqrt{5}}{2}$ or its inverse $\alpha^{-1} = \frac{-1+\sqrt{5}}{2}$ and are related to the alternate terms of the Fibonacci sequence.

Corollary 4.2. *The integral sequences for $A = 1$ have starting values x_1, x_2 which are consecutive terms in one of the four bi-infinite sequences listed here:*

$$\ldots, -89, 34, -13, 5, -2, 1, -1, 2, -5, 13, -34, 89, \ldots,$$

$$\ldots, 89, -34, 13, -5, 2, -1, 1, -2, 5, -13, 34, -89, \ldots,$$

$$\ldots, -89, -34, -13, -5, -2, -1, -1, -2, -5, -13, -34, -89, \ldots,$$
$$\ldots, 89, 34, 13, 5, 2, 1, 1, 2, 5, 13, 34, 89, \ldots.$$

Proof. From the theorem we need to consider $\mu = \pm 3$ and the solutions to $X^2 - 5Y^2 = -4$. The solutions are the odd powers of $\pm\alpha$, $\pm\alpha^{-1}$ which give the sequences listed above. □

References

[1] S. Fomin and A. Zelevinsky, *The Laurent phenomenon*, Advances in App. Math., **28** (2002), 119-144.
[2] W. Sierpinski, *Elementary Theory of Numbers*, Warsaw, 1964.

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