EVERY POSITIVE K-BONACCI-LIKE SEQUENCE EVENTUALLY AGREES WITH A ROW OF THE K-ZECKENDORF ARRAY

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ABSTRACT. For $k \geq 2$, a fixed integer, we work with the k-bonacci sequence, $\{X_n\}$, a kth order generalization of the Fibonacci numbers, and their use in a Zeckendorf representation of positive integers. We extend Zeckendorf representations using $\{X_n \mid n \in \mathbb{Z}\}$ and show that every sequence of positive integers satisfying the k-bonacci recurrence eventually agrees with a row of the k-Zeckendorf array.

Throughout this paper, $k \geq 2$ is a fixed integer.

Definition 1. The *k-bonacci sequence* $\{X_n\}$ is given by the recurrence

$$\begin{aligned} X_n &= 0 && \text{for } -k+2 \leq n \leq 0, \\ X_1 &= 1, \\ X_n &= \sum_{i=1}^{k} X_{n-i} && \text{for all } n \in \mathbb{Z}. \end{aligned}$$

When $k = 2$, $\{X_n\}$ is the Fibonacci sequence, when $k = 3$ the tribonacci sequence, and so on. We have deliberately used $\mathbb{Z}$ as the domain of subscripts for $\{X_n\}$.

Our purpose herein is to generalize the following well-known theorem [6] (see also [4]). (Strictly speaking, Zeckendorf's Theorem applies to the Fibonacci numbers ($k = 2$), but the proof via greedy change-making applies without change to k-bonacci numbers.)

Theorem 1. Zeckendorf's Theorem. *Every nonnegative number, n, is a unique sum of distinct k-bonacci numbers:*

$$n = \sum_{i \geq 2} c_i X_i$$

such that $c_i \in \{0, 1\}$ for all i, and no string of k consecutive c_i's are equal to 1.

Definition 2. A *coefficient list* is a function $d : \mathbb{Z} \to \mathbb{Z}^{\geq 0}$ with finite support (i.e., $d(i) > 0$ for only a finite number of i). In case $\prod_{i=1}^{k} d(j+i) = 0$ for all j, d is called *weakly Zeckendorf*. If, in addition, $d(i) \in \{0, 1\}$ for all i, d is called *Zeckendorf.*

Definition 3. The *Zeckendorf value of a coefficient list* is $Z(d) = \sum_{i \in \mathbb{Z}} d(i) X_i$. This is defined whether or not d is (weakly) Zeckendorf.

It is easy to see that for any integer n, there are an infinite number of coefficient lists d such that $Z(d) = n$, even an infinite number of Zeckendorf coefficient lists.

Definition 4. The *sum of a coefficient list* is $\sum_{i \in \mathbb{Z}} d(i)$.

Definition 5. The *shift*, Sd, of a coefficient list is the coefficient list satisfying $Sd(i) = d(i-1)$. We extend this notion to include the shift of a positive integer n. If n has the unique Zeckendorf representation $n = \sum_{i=2}^{m} d(i) X_i$, then $S(n) = Z(Sd)$.

Here is how we generalize Zeckendorf's Theorem to deal with $\{X_n\}$ using subscripts in $\mathbb{Z}$.

Theorem 2. *For any sequence of k non-negative integers $(z_0, z_1, \ldots, z_{k-1})$ there is a unique Zeckendorf coefficient list d such that $z_i = Z(S^i d)$.*

The proof requires a small amount of machinery.

Definition 6. The function $r : \mathbb{Z} \to \{-1, 0, 1\}$ expresses the k-bonacci defining rule:

$$r(i) = \begin{cases} -1 & -k \leq i < 0, \\ 1 & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 7. Following the concepts of grade-school arithmetic, the *carry into position j* of a coefficient list, $C_j d$, is defined by

$$C_j d(i) = d(i) + r(i - j).$$

The carry is called *proper* in case $C_j d$ is a coefficient list (i.e., its range is in $\mathbb{Z}^{\geq 0}$), and is *highly proper* if, in addition, $d(j) = 0$. The *borrow from position j*, $B_j d$, is defined by

$$B_j d(i) = d(i) - r(i - j).$$

The borrow is called *proper* in case $B_j d$ is a coefficient list (i.e., $d(j) > 0$ so the range of $B_j d$ is in $\mathbb{Z}^{\geq 0}$).

These definitions have some straightforward consequences:

(1) Carrying and borrowing are inverses of each other: $B_j d = C_j^{-1} d$.
(2) Carrying and borrowing do not change the Zeckendorf value of a coefficient list: $Z(d) = Z(C_j d) = Z(B_j d)$.
(3) Carrying and borrowing commute with shifting: $B_{j+1} S d = S B_j d$ and $C_{j+1} S d = S C_j d$.

Definition 8. Call two coefficient lists, d and d', *equivalent* if they are related by a sequence of carries and borrows: $d' = C_{j_m}^{\pm 1} \cdots C_{j_1}^{\pm 1} d$.

The following lemma allows us to prove Theorem 2.

Lemma 3. *Every coefficient list is equivalent to a Zeckendorf coefficient list.*

Proof. First, we show that every coefficient list is equivalent to a weakly Zeckendorf coefficient list. Suppose $d(i+1), \ldots, d(i+k)$ are k positive values. The equivalent coefficient list $C_{i+k+1} d$ has a smaller sum. Our claim follows by induction on the sum of the coefficient list. (Here and in the following step, we use the principle of mathematical induction that a decreasing sequence of non-negative integers must terminate.)

Next, we demonstrate that every weakly Zeckendorf coefficient list is equivalent to a Zeckendorf coefficient list. If d is weakly Zeckendorf, let i be the largest value for which $d(i) > 1$ and $d(j), \ldots, d(i), \ldots, d(j')$ be positive values with $j \leq i \leq j'$, $j' - j < k - 1$, and $d(j-1) = d(j'+1) = 0$. Let the *bad sum* of the coefficient list of d be $\beta(d) = \sum_{n \leq j'} d(n)$. Produce the equivalent d' by borrowing from position i and highly proper carrying into position $j'+1$ (so d and d' have the same sum): $d' = C_{j'+1} B_i d$. For $n > j'+1$, $d'(n) = d(n)$, but $d'(j'+1) = 1$, so $\beta(d') < \beta(d)$, and the lemma follows by induction on the bad sum of the coefficient list. (d' may have to be modified by some highly proper carries to convert it to weakly Zeckendorf, but that will not increase the sum of the coefficient list. In fact, it will decrease the sum of the coefficient list and the bad sum of the modified coefficient list will not increase.) □

At this point we will give an example to demonstrate the concept of the bad sum of a coefficient list and that every weakly Zeckendorf coefficient list is equivalent to a Zeckendorf coefficient list. We will use the 3-bonacci (tribonacci) sequence

i	0	1	2	3	4	5	6	7	8	9	10	11	12
X_i	0	1	1	2	4	7	13	24	44	81	149	274	504

Consider the weakly Zeckendorf coefficient list

i	0	1	2	3	4
$d(i)$	1	4	0	2	1

The bad sum is 8. Transforming the d coefficient list by borrowing B_3 and carrying C_5 we obtain the equivalent d coefficient list

i	0	1	2	3	4	5
$d(i)$	2	5	0	0	0	1

The bad sum is now 7. Again, transforming the d coefficient list by borrowing B_1 and carrying C_2 we obtain the equivalent d coefficient list

i	-2	-1	0	1	2	3	4	5
$d(i)$	1	0	2	3	1	0	0	1

Since d is no longer weakly Zeckendorf, we need to do the carry C_3 to restore it to the weakly Zeckendorf form. We thus have the equivalent d coefficient list

i	-2	-1	0	1	2	3	4	5
$d(i)$	1	0	1	2	0	1	0	1

The bad sum is now 4. We now borrow B_1 and carry C_2 to obtain the equivalent d coefficient list

i	-2	-1	0	1	2	3	4	5
$d(i)$	2	0	1	0	1	1	0	1

The bad sum is now 2. We finally borrow using B_{-2} and carry using C_{-1} to obtain the Zeckendorf coefficient list d.

i	-5	-4	-3	-2	-1	0	1	2	3	4	5
$d(i)$	1	0	0	0	1	1	0	1	1	0	1

We may now prove Theorem 2.

Proof. Because of Lemma 3, we need only display (or show how to find) an unrestricted coefficient list d such that $Z(S^i d)$, for $0 \leq i < k$, can take on arbitrary specified values.

Let d be a coefficient list with support in the interval of size k: $[-k+2, 1]$. So $Z(S^i d) = \sum_{j=0}^{i} X_{i+1-j} d(1-j)$. Because of the simple form of the values $X_{-k+2}, \ldots, X_1$, it is straightforward to determine d to achieve any desired values for the $Z(S^i d)$. $\square$

If we are dealing with the Fibonacci numbers ($k = 2$), the support of d is $[0, 1]$, and $Z(d) = d(1)$ and $Z(Sd) = d(0) + d(1)$. Appropriate choices of $d(0)$ and $d(1)$ can give any desired values for $Z(d)$ and $Z(Sd)$.

If we are dealing with the tribonacci numbers ($k = 3$), the support of d is $[-1, 0, 1]$, and $Z(d) = d(1)$, $Z(Sd) = d(0) + d(1)$, and $Z(S^2d) = d(-1) + d(0) + 2d(1)$. Appropriate choices of d can give any desired values for $Z(d), Z(Sd), Z(S^2d)$.

We continue to work with k-bonacci numbers for some fixed $k \geq 2$.

Definition 9. The *k-Zeckendorf array* is an infinite table of numbers $\{A_{ij}\}$ where the subscripts i and j range over the positive integers. The first row consists of the k-bonacci numbers: $A_{1j} = X_{j+1}$. The first value in row i for $i > 1$, A_{i1} is the smallest number not appearing in any previous row. Then, subsequent values in each row follow the rule: $A_{i,j+1} = S(A_{ij})$.

Several facts are known about these arrays [1, 2, 3]:

- Every positive integer occurs exactly once in the array.
- The first column of the array consists of the integers n such that the coefficient of X_2 in their k-Zeckendorf representation is 1.

Definition 10. A *positive k-bonacci-like sequence* $\{W_n\}$ is a sequence of positive integers that obeys the recurrence of Definition 1: $W_n = \sum_{i=1}^{k} W_{n-i}$ for all $n \in \mathbb{Z}$.

Theorem 4. *Every positive k-bonacci-like sequence, after some position, occurs as a row of the k-Zeckendorf array.*

Proof. Let d be the Zeckendorf coefficient list satisfying $Z(S^i d) = W_i$ for $0 \leq i < k$. (Note that $W_i = Z(S^i d)$ for all i.) Let $m = \min\{n \mid d(n) = 1\}$. There are two cases.

Case 1. $m \geq 2$. Let $z = Z(d)$. Since $z \geq 1$, $z = A_{j,k}$ for some $j, k \geq 1$. Therefore, $W_i = A_{j,k+i}$ for $i \geq 0$.

Case 2. $m < 2$. Let $z = Z(S^{-m+2}d)$. Since $z \geq 1$, $z = A_{j,1}$ for some $j \geq 1$. Thus, $W_{-m+2+i} = A_{j,1+i}$ for $i \geq 0$.

□

David R. Morrison [5] established the Fibonacci case $k = 2$ of Theorem 4 using Wythoff pairs.

We illustrate the proof of Theorem 4 with two examples. For both examples we will again use the 3-bonacci (tribonacci) sequence

i	0	1	2	3	4	5	6	7	8	9	10	11	12
X_i	0	1	1	2	4	7	13	24	44	81	149	274	504

and the 3-Zeckendorf array A given by

1	2	4	7	13	24	44	81	149	274	504	⋯
3	6	11	20	37	68	125	230	423	778	1431	
5	9	17	31	57	105	193	355	653	1201	2209	
8	15	28	51	94	173	318	585	1076	1979	3640	
10	19	35	64	118	217	399	734	1350	2483	4567	
12	22	41	75	138	254	467	859	1580	2906	5345	
14	26	48	88	162	298	548	1008	1854	3410	6272	
16	30	55	101	186	342	629	1157	2128	3914	7199	
18	33	61	112	206	379	697	1282	2358	4337	7977	
⋮											

Consider the positive 3-bonacci-like sequence

i	0	1	2	3	4	5	6	7
W_i	1	3	4	8	15	27	50	92

To find a coefficient list d for $\{W_n\}$ we solve the system of equations

$$d(1) = 1$$
$$d(0) + d(1) = 3$$
$$d(-1) + d(0) + 2d(1) = 4$$

to obtain the coefficient list

i	-1	0	1
$d(i)$	0	2	1

This coefficient sequence is already weakly Zeckendorf. Using the second half of the proof of Lemma 3, the bad sum is 3. We produce the equivalent d coefficient list by the borrow B_0 followed by the carry C_2 to obtain the d coefficient list

i	-3	-2	-1	0	1	2
$d(i)$	1	1	0	0	0	1

Using the recurrence relation for the tribonacci sequence we see that $X_{-3} = -1$ and $X_{-2} = 1$, so we can show that $Z(S^i d) = W_i$. Since $m = -3$, $z = Z(S^5 d) = 27$. But $27 = A_{13,1}$. So $W_{5+i} = A_{13,1+i}$ for $i \geq 0$.

Next, consider the positive 3-bonacci-like sequence

i	0	1	2	3	4	5	6	7
W_i	6	11	20	37	68	125	230	423

To find a coefficient list d for $\{W_n\}$ we solve the system of equations

$$d(1) = 6$$
$$d(0) + d(1) = 11$$
$$d(-1) + d(0) + 2d(1) = 20$$

to obtain the coefficient list

i	-1	0	1
$d(i)$	3	5	6

Now we need the first half of the proof of Lemma 3. Using a sequence of carries, each new coefficient list will have a smaller sum than the previous coefficient list. First, we do the carry C_2 to obtain the equivalent d coefficient list

i	-1	0	1	2
$d(i)$	2	4	5	1

Next, we do the carry C_3 to obtain the d coefficient list

i	-1	0	1	2	3
$d(i)$	2	3	4	0	1

Next we do the carry C_2 to obtain the d coefficient list

i	-1	0	1	2	3
$d(i)$	1	2	3	1	1

Next, we do the carry C_4 we obtain the d coefficient list

i	-1	0	1	2	3	4
$d(i)$	1	2	2	0	0	1

Next, we do the carry C_2 to obtain the d coefficient list

i	0	1	2	3	4
$d(i)$	1	1	1	0	1

Finally, doing the carry C_3, we obtain the d coefficient list

i	0	1	2	3	4
$d(i)$	0	0	0	1	1

Since $m = 3$, $z = Z(d) = 6$. But $6 = A_{2,2}$. Therefore, $W_i = A_{2,2+i}$ for $i \geq 0$.

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