FORMULAS FOR FIBONOMIAL SUMS WITH GENERALIZED FIBONACCI AND LUCAS COEFFICIENTS

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Abstract. We consider certain Fibonomial sums with generalized Fibonacci and Lucas numbers coefficients and compute them explicitly. Some corollaries are also presented. The technique is to rewrite everything in terms of a variable q, and then to use Rothe's identity from classical q-calculus.

1. Introduction

Define the second order linear sequences $\{U_n\}$ and $\{V_n\}$ for $n \geq 2$ by

$$U_n = pU_{n-1} + U_{n-2}, \quad U_0 = 0,\ U_1 = 1,$$
$$V_n = pV_{n-1} + V_{n-2}, \quad V_0 = 2,\ V_1 = p.$$

For $n \geq k \geq 1$, define the generalized Fibonomial coefficient by

$$\left\{ {n \atop k} \right\}_U := \frac{U_1 U_2 \cdots U_n}{(U_1 U_2 \cdots U_k)(U_1 U_2 \cdots U_{n-k})}$$

with $\left\{ {n \atop 0} \right\}_U = \left\{ {n \atop n} \right\}_U = 1$. When $p = 1$, we obtain the usual Fibonomial coefficient, denoted by $\left\{ {n \atop k} \right\}_F$.

In this paper, we will present two sets of 4 identities each, which are presented in the notion of $\left\{ {n \atop k} \right\}_U$.

Our approach will be as follows. We will use the Binet forms

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \alpha^{n-1} \frac{1 - q^n}{1 - q} \quad \text{and} \quad V_n = \alpha^n + \beta^n = \alpha^n (1 + q^n)$$

with $q = \beta/\alpha = -\alpha^{-2}$, so that $\alpha = \mathbf{i}/\sqrt{q}$ where $\alpha, \beta = \left(p \pm \sqrt{p^2+4}\right)/2$.

Throughout this paper we will use the following notations: the q-Pochhammer symbol $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$ and the Gaussian q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

As stated, this defines the q-Pochhammer symbol only for nonnegative integers. It is extended as follows: one forms the infinite product

$$(x;q)_\infty := \prod_{j \geq 0} (1 - xq^j),$$

and notices that

$$(x;q)_n = \frac{(x;q)_\infty}{(xq^n;q)_\infty},$$

thanks to cancellations. The righthand side, however, makes sense for any $n \in \mathbb{C}$; in particular for negative integers we have

$$(x;q)_{-n} = \frac{(x;q)_\infty}{(xq^{-n};q)_\infty} = \frac{(x;q)_\infty}{(xq^{-n};q)_n (x;q)_\infty} = \frac{1}{(xq^{-n};q)_n}.$$

The link between the generalized Fibonomial and Gaussian q-binomial coefficients is

$$\left\{ {n \atop k} \right\}_U = \alpha^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \quad \text{with} \quad q = -\alpha^{-2}.$$

We recall that one version of the *Cauchy Binomial Theorem* is given by

$$\sum_{k=0}^{n} q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = \prod_{k=1}^{n} \left(1 + xq^k\right),$$

and *Rothe's* Formula [1] is

$$\sum_{k=0}^{n} (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k = (x;q)_n = \prod_{k=0}^{n-1} \left(1 - xq^k\right).$$

All the identities we will derive hold for general q, and results about generalized Fibonacci and Lucas numbers come out as corollaries for the special choice of q. We will frequently denote $\left\{ {n \atop k} \right\}_U$ by $\left\{ {n \atop k} \right\}$.

We shall consider some Fibonomial sums with generalized Fibonacci and Lucas numbers as coefficients, and then we compute these sums by using Rothe's Formula after having converted them into forms involving the Gaussian q-binomial coefficients. Some special cases of these sums are also given as corollaries.

Here are our main results.

Theorem 1. *If n and m are both nonnegative integers, then*

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} U_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \left\{ {2m-1 \atop 2k-1} \right\} U_{(4k-2)n},$$

$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} U_{2mk} = P_{n,m} \sum_{k=0}^{m} \left\{ {2m \atop 2k} \right\} U_{(2n+1)2k},$$

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} (-1)^k U_{(2m-1)k} = P_{n,m} \sum_{k=0}^{m-1} \left\{ {2m-1 \atop 2k} \right\} U_{4kn},$$

$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} (-1)^k U_{2mk} = -P_{n,m} \sum_{k=1}^{m} \left\{ {2m \atop 2k-1} \right\} U_{(2n+1)(2k-1)},$$

where

$$P_{n,m} = \begin{cases} \prod\limits_{k=0}^{n-m} V_{2k} & \text{if } n \geq m, \\ \prod\limits_{k=1}^{m-n-1} V_{2k}^{-1} & \text{if } n < m. \end{cases}$$

Theorem 2. *If n and m are both nonnegative integers, then*

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} V_{(2m-1)k} = P_{n,m} \sum_{k=1}^{m} \left\{ {2m-1 \atop 2k-1} \right\} V_{(4k-2)n},$$
$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} V_{2mk} = P_{n,m} \sum_{k=0}^{m} \left\{ {2m \atop 2k} \right\} V_{(2n+1)2k},$$
$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} (-1)^k V_{(2m-1)k} = P_{n,m} \sum_{k=0}^{m-1} \left\{ {2m-1 \atop 2k} \right\} V_{4kn},$$
$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} (-1)^k V_{2mk} = -P_{n,m} \sum_{k=1}^{m} \left\{ {2m \atop 2k-1} \right\} V_{(2n+1)(2k-1)},$$

where $P_{n,m}$ is defined as before.

For example, when $m = n$ in Theorem 1, we have the following identities:

$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} U_{(2n-1)k} = 2 \sum_{k=1}^{n} \left\{ {2n-1 \atop 2k-1} \right\} U_{(4k-2)n},$$
$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} U_{2nk} = 2 \sum_{k=0}^{n} \left\{ {2n \atop 2k} \right\} U_{(2n+1)2k},$$
$$\sum_{k=0}^{2n} \left\{ {2n \atop k} \right\} (-1)^k U_{(2n-1)k} = 2 \sum_{k=0}^{n-1} \left\{ {2n-1 \atop 2k} \right\} U_{4kn},$$
$$\sum_{k=0}^{2n+1} \left\{ {2n+1 \atop k} \right\} (-1)^k U_{2nk} = -2 \sum_{k=1}^{n} \left\{ {2n \atop 2k-1} \right\} U_{(2n+1)(2k-1)}.$$

2. Proofs

We will only prove the first formula of the first theorem since all the other verifications are very similar. Since the argument is quite involved, we break it down for the reader's convenience:

(1) Both sides are translated into q-notation.
(2) Both sides are "computed" using Rothe's Theorem, introducing complex entries. This leads to 4 terms on each side.
(3) These terms are matched with each other, which leaves us with 4 identities that are elementary (but tedious) rearrangements. However, the matching of the 4 pairs of formulas depends on the parities of m and n.

First suppose that $n \geq m$. We rewrite $P_{n,m}$ in terms of q-binomial coefficients:

$$\begin{aligned}
P_{n,m} &= \prod_{k=0}^{n-m} V_{2k} = \prod_{k=0}^{n-m} (\alpha^{2k} + \beta^{2k}) \\
&= \alpha^{(n-m)(n-m+1)} \prod_{k=0}^{n-m} (1 + q^{2k}) = 2\alpha^{(n-m)(n-m+1)} (-q^2; q^2)_{n-m} \\
&= 2(-q)^{-\binom{n-m+1}{2}} (-q^2; q^2)_{n-m}.
\end{aligned}$$

This formula holds for $n < m$ as well, with the usual extension of $(q; q)_n$ to arbitrary n.

Similarly, the first formula takes the following form in terms of q-binomial coefficients:

$$\begin{aligned}
\sum_{k=0}^{2n} \frac{\alpha^{(2m-1)k} - \beta^{(2m-1)k}}{\alpha - \beta} \alpha^{k(2n-k)} \begin{bmatrix} 2n \\ k \end{bmatrix}_q & \\
= 2\alpha^{(n-m)(n-m+1)} (-q^2; q^2)_{n-m} & \\
\times \sum_{k=1}^{m} \frac{\alpha^{(4k-2)n} - \beta^{(4k-2)n}}{\alpha - \beta} \alpha^{(2k-1)(2m-1-2k+1)} \begin{bmatrix} 2m-1 \\ 2k-1 \end{bmatrix}_q &,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\sum_{k=0}^{2n} [1 - q^{(2m-1)k}] \alpha^{2(m+n)k - 2\binom{k+1}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q & \\
= 2\alpha^{(n-m)(n-m+1)-2(m+n)} (-q^2; q^2)_{n-m} & \\
\times \sum_{k=1}^{m} [1 - q^{(4k-2)n}] \alpha^{4k(m+n)-2k(2k-1)} \begin{bmatrix} 2m-1 \\ 2k-1 \end{bmatrix}_q &,
\end{aligned}$$

and to

$$\begin{aligned}
\sum_{k=0}^{2n} [1 - q^{(2m-1)k}] (-q)^{-(m+n)k + \binom{k+1}{2}} \begin{bmatrix} 2n \\ k \end{bmatrix}_q & \\
= 2(-q)^{-\binom{n-m+1}{2} + (m+n)} (-q^2; q^2)_{n-m} & \\
\times \sum_{k=1}^{m} [1 - q^{(4k-2)n}] (-1)^k q^{-2k(m+n)+k(2k-1)} \begin{bmatrix} 2m-1 \\ 2k-1 \end{bmatrix}_q &.
\end{aligned}$$

We are going to prove this form.

If we denote the left and right hand sides of this equation by L and R, respectively, then L is the sum of the following two parts:

$$\begin{aligned}
L_1 &= \sum_{k=0}^{2n}(-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q \\
&= \sum_{k=0}^{2n}(-1)^{-(m+n-\frac{1}{2})k+\frac{k^2}{2}}q^{-(m+n-1)k+\binom{k}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q \\
&= \sum_{k=0}^{2n}(-1)^{-(m+n-\frac{1}{2})k}q^{-(m+n-1)k+\binom{k}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q\left[\frac{1+\mathbf{i}}{2}+\frac{1-\mathbf{i}}{2}(-1)^k\right] \\
&= \frac{1+\mathbf{i}}{2}\sum_{k=0}^{2n}(-1)^{-(m+n-\frac{1}{2})k}q^{-(m+n-1)k+\binom{k}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q \\
&\quad + \frac{1-\mathbf{i}}{2}\sum_{k=0}^{2n}(-1)^{-(m+n+\frac{1}{2})k}q^{-(m+n-1)k+\binom{k}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q \\
&= \frac{1+\mathbf{i}}{2}(\mathbf{i}(-q)^{-(m+n-1)};q)_{2n}+\frac{1-\mathbf{i}}{2}(-\mathbf{i}(-q)^{-(m+n-1)};q)_{2n}
\end{aligned}$$

and

$$\begin{aligned}
L_2 &= -\sum_{k=0}^{2n}q^{(2m-1)k}(-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q \\
&= -\sum_{k=0}^{2n}(-1)^{(m-n+\frac{1}{2})k+\frac{k^2}{2}}q^{(m-n)k+\binom{k}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q \\
&= -\sum_{k=0}^{2n}(-1)^{(m-n+\frac{1}{2})k}q^{(m-n)k+\binom{k}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q\left[\frac{1+\mathbf{i}}{2}+\frac{1-\mathbf{i}}{2}(-1)^k\right] \\
&= -\frac{1+\mathbf{i}}{2}\sum_{k=0}^{2n}(-1)^{(m-n+\frac{1}{2})k}q^{(m-n)k+\binom{k}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q \\
&\quad - \frac{1-\mathbf{i}}{2}\sum_{k=0}^{2n}(-1)^{(m-n-\frac{1}{2})k}q^{(m-n)k+\binom{k}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q \\
&= -\frac{1+\mathbf{i}}{2}(-\mathbf{i}(-q)^{m-n};q)_{2n}-\frac{1-\mathbf{i}}{2}(\mathbf{i}(-q)^{m-n};q)_{2n}.
\end{aligned}$$

The evaluations of the sums have been done with Rothe's formula.

By combining the two parts above we write L as

$$\begin{aligned}
&\frac{1+\mathbf{i}}{2}(\mathbf{i}(-q)^{-(m+n-1)};q)_{2n}+\frac{1-\mathbf{i}}{2}(-\mathbf{i}(-q)^{-(m+n-1)};q)_{2n} \\
&\qquad -\frac{1+\mathbf{i}}{2}(-\mathbf{i}(-q)^{m-n};q)_{2n}-\frac{1-\mathbf{i}}{2}(\mathbf{i}(-q)^{m-n};q)_{2n} \\
&= L_a+L_b+L_c+L_d.
\end{aligned}$$

Let

$$\begin{aligned}
R_1 &= \sum_{k=1}^{m}[1-q^{(4k-2)n}](-1)^k q^{-2k(m+n)+k(2k-1)} \begin{bmatrix} 2m-1 \\ 2k-1 \end{bmatrix}_q \\
&= \mathbf{i}q^{-(m+n)} \sum_{k=0}^{2m-1}[1-q^{2kn}]\mathbf{i}^k q^{-k(m+n-1)+\binom{k}{2}} \begin{bmatrix} 2m-1 \\ k \end{bmatrix}_q \frac{1-(-1)^k}{2} \\
&= \frac{1}{2}\mathbf{i}q^{-(m+n)} \sum_{k=0}^{2m-1}[1-q^{2kn}]\mathbf{i}^k q^{-k(m+n-1)+\binom{k}{2}} \begin{bmatrix} 2m-1 \\ k \end{bmatrix}_q \\
&\quad - \frac{1}{2}\mathbf{i}q^{-(m+n)} \sum_{k=0}^{2m-1}[1-q^{2kn}](-\mathbf{i})^k q^{-k(m+n-1)+\binom{k}{2}} \begin{bmatrix} 2m-1 \\ k \end{bmatrix}_q \\
&= \frac{1}{2}\mathbf{i}q^{-(m+n)}(-\mathbf{i}q^{-m-n+1};q)_{2m-1} - \frac{1}{2}\mathbf{i}q^{-(m+n)}(-\mathbf{i}q^{-m+n+1};q)_{2m-1} \\
&\quad - \frac{1}{2}\mathbf{i}q^{-(m+n)}(\mathbf{i}q^{-m-n+1};q)_{2m-1} + \frac{1}{2}\mathbf{i}q^{-(m+n)}(\mathbf{i}q^{-m+n+1};q)_{2m-1}.
\end{aligned}$$

Again, Rothe's formula has been used for simplification.

In order to form the right hand side R, the last expression must be multiplied by

$$2(-q)^{-\binom{n-m+1}{2}+(m+n)}(-q^2;q^2)_{n-m}.$$

Thus R takes the form:

$$\begin{aligned}
R &= \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(-\mathbf{i}q^{-m-n+1};q)_{2m-1} \\
&\quad - \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(-\mathbf{i}q^{-m+n+1};q)_{2m-1} \\
&\quad - \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1} \\
&\quad + \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m+n+1};q)_{2m-1} \\
&= R_a + R_b + R_c + R_d.
\end{aligned}$$

We will show that for $m \equiv n \pmod 2$,

$$L_a = R_a, \quad L_b = R_c, \quad L_c = R_b, \quad L_d = R_d,$$

and for $m \not\equiv n \pmod 2$,

$$L_a = R_c, \quad L_b = R_a, \quad L_c = R_d, \quad L_d = R_b.$$

We start with the instance $m \equiv n \pmod 2$, and first show that $L_a = R_a$. If we rearrange both sides of it, the claimed equality becomes

$$\frac{1+\mathbf{i}}{2}(\mathbf{i}(-q)^{-(m+n-1)};q)_{2n} = \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(-\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1+\mathbf{i}}{2}(-\mathbf{i}q^{-m-n+1};q)_{2n} = \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(-\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1+\mathbf{i}}{2}(-\mathbf{i}q^{m-n};q)_{2n-2m+1} = \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}.$$

In order to show the last equality, we consider two cases. For even N, by rearranging both sides of it we get

$$\frac{1+\mathbf{i}}{2}(-\mathbf{i}q^{-N};q)_{2N+1} = \mathbf{i}(-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N,$$

or

$$\frac{1+\mathbf{i}}{2}\prod_{k=0}^{2N}(1+\mathbf{i}q^{-N+k}) = \mathbf{i}(-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N,$$

or

$$\frac{1+\mathbf{i}}{2}\prod_{k=1}^{N}(1+\mathbf{i}q^{-k})\prod_{k=1}^{N}(1+\mathbf{i}q^{k})(1+\mathbf{i}) = \mathbf{i}(-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N,$$

or

$$\prod_{k=1}^{N}(1+\mathbf{i}q^{-k})\prod_{k=1}^{N}(1+\mathbf{i}q^{k}) = (-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N,$$

or

$$\prod_{k=1}^{N}\mathbf{i}(q^{-k}+q^{k}) = (-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N,$$

or

$$\mathbf{i}^N q^{-\binom{N+1}{2}}\prod_{k=1}^{N}(1+q^{2k}) = (-q)^{-\binom{N+1}{2}}(-q^2;q^2)_N,$$

or

$$\mathbf{i}^N = (-1)^{N/2} = (-1)^{-(N+1)\frac{N}{2}},$$

as claimed.

Now we prove the second claim $L_b = R_c$. By rearranging both sides of it, we get

$$\frac{1-\mathbf{i}}{2}(-\mathbf{i}(-q)^{-(m+n-1)};q)_{2n} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1-\mathbf{i}}{2}(\mathbf{i}q^{-m-n+1};q)_{2n} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1-\mathbf{i}}{2}(\mathbf{i}q^{m-n};q)_{2n-2m+1} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$\frac{1-\mathbf{i}}{2}\prod_{k=0}^{2n-2m}(1-\mathbf{i}q^{m-n+k}) = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$\frac{1-\mathbf{i}}{2}\prod_{k=1}^{n-m}(1-\mathbf{i}q^{-k})(1-\mathbf{i}q^{k})\cdot(1-\mathbf{i}) = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$(-\mathbf{i})^{n-m}\prod_{k=1}^{n-m}q^{-k}(1+q^{2k}) = (-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

which becomes

$$\mathbf{i}^{n-m} = (-1)^{-\binom{n-m+1}{2}},$$

as claimed.

We note that the other cases (for $m \equiv n \pmod 2$) can be done similarly.

Now we consider the case $L_a = R_c$ if $m \not\equiv n \pmod 2$. By simplifying both sides of the claimed equality step by step, we get

$$\frac{1+\mathbf{i}}{2}(\mathbf{i}(-q)^{-(m+n-1)};q)_{2n} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1+\mathbf{i}}{2}(\mathbf{i}q^{-m-n+1};q)_{2n} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m}(\mathbf{i}q^{-m-n+1};q)_{2m-1},$$

or

$$\frac{1+\mathbf{i}}{2}(\mathbf{i}q^{m-n};q)_{2n-2m+1} = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$\frac{1+\mathbf{i}}{2}\prod_{k=0}^{2n-2m}(1-\mathbf{i}q^{m-n+k}) = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$\frac{1+\mathbf{i}}{2}\prod_{k=1}^{n-m}(1-\mathbf{i}q^{-k})(1-\mathbf{i}q^{k})\cdot(1-\mathbf{i}) = -\mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$\mathbf{i}^{n-m}\prod_{k=1}^{n-m}q^{-k}(1+q^{2k}) = \mathbf{i}(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m},$$

or

$$\mathbf{i}^{n-m} = \mathbf{i}(-1)^{-\binom{n-m+1}{2}},$$

which is true as claimed.

The other cases (for $m \not\equiv n \pmod 2$) can be done similarly.

The arguments hold for $n < m$ as well.

As announced, the remaining 3 cases of the first and the 4 cases of the second theorem are completely analogous and therefore omitted to save space (and the patience of the gentle reader).

3. The Identities in q-notation

For the reader's convenience, here is the complete list of q-binomial versions of the identities given in Theorem 1 and Theorem 2. Let n and m both be nonnegative integers,

$$\sum_{k=0}^{2n}[1-q^{(2m-1)k}](-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q$$
$$= P_{n,m}\sum_{k=1}^{m}[1-q^{(4k-2)n}](-q)^{-(2k-1)(m+n-k)}\begin{bmatrix}2m-1\\2k-1\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1}[1-q^{2mk}](-q)^{-(m+n)k+\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_q$$
$$= P_{n,m}\sum_{k=0}^{m}[1-q^{2k(2n+1)}](-q)^{k(2k-2m-2n-1)}\begin{bmatrix}2m\\2k\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n}(-1)^k[1-q^{(2m-1)k}](-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q$$
$$=P_{n,m}\sum_{k=0}^{m-1}[1-q^{4kn}](-q)^{k(2k-2m-2n+1)}\begin{bmatrix}2m-1\\2k\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1}(-1)^k[1-q^{2mk}](-q)^{-(m+n)k+\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_q$$
$$=-P_{n,m}\sum_{k=1}^{m}[1-q^{(2k-1)(2n+1)}](-q)^{(2k-1)(k-m-n-1)}\begin{bmatrix}2m\\2k-1\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n}[1+q^{(2m-1)k}](-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q$$
$$=P_{n,m}\sum_{k=1}^{m}[1+q^{(4k-2)n}](-q)^{-(2k-1)(m+n-k)}\begin{bmatrix}2m-1\\2k-1\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1}[1+q^{2mk}](-q)^{-(m+n)k+\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_q$$
$$=P_{n,m}\sum_{k=0}^{m}[1+q^{2k(2n+1)}](-q)^{k(2k-2m-2n-1)}\begin{bmatrix}2m\\2k\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n}(-1)^k[1+q^{(2m-1)k}](-q)^{-(m+n)k+\binom{k+1}{2}}\begin{bmatrix}2n\\k\end{bmatrix}_q$$
$$=P_{n,m}\sum_{k=0}^{m-1}[1+q^{4kn}](-q)^{k(2k-2m-2n+1)}\begin{bmatrix}2m-1\\2k\end{bmatrix}_q,$$

$$\sum_{k=0}^{2n+1}(-1)^k[1+q^{2mk}](-q)^{-(m+n)k+\binom{k}{2}}\begin{bmatrix}2n+1\\k\end{bmatrix}_q$$
$$=-P_{n,m}\sum_{k=1}^{m}[1+q^{(2k-1)(2n+1)}](-q)^{(2k-1)(k-m-n-1)}\begin{bmatrix}2m\\2k-1\end{bmatrix}_q,$$

where

$$P_{n,m}=\begin{cases}2(-q)^{-\binom{n-m+1}{2}}(-q^2;q^2)_{n-m} & \text{if } n\ge m,\\ (-q)^{\binom{m-n}{2}}(-q^2;q^2)^{-1}_{m-n-1} & \text{if } n<m.\end{cases}$$

Remark. *It is not necessary to split the definition of $P_{n,m}$, as the first alternative would work in both cases, but it is more convenient as given.*

4. Conclusion

The coefficients
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$
are only defined for n, k nonnegative integers. However, as explained in the Introduction, we could allow n, k, $n-k$ to be anything. In order to give sense to
$$\sum_{k=0}^{n} a_k,$$
we could proceed like this:
$$\sum_{k=0}^{n} a_k = \sum_{k\geq 0} a_k - \sum_{k\geq n+1} a_k = \sum_{k\geq 0} a_k - \sum_{k\geq 0} a_{n+k+1},$$
and this works well if the coefficients a_k make sense for arbitrary indices and the series converge. However, if n is a negative integer, then
$$\sum_{k\geq 0} a_k - \sum_{k\geq 0} a_{n+k+1} = -\sum_{k=n+1}^{-1} a_k,$$
and this can be used even without convergence issues.

We believe that our formulas – with these extended definitions – hold as well for a larger range of values for m and n. However, we have not checked details here.

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References

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