LUCAS' HYPERBOLAS FOR FIBONACCI VECTORS

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ABSTRACT. E. Lucas showed that every point in the plane whose coordinates are consecutive Fibonacci numbers lies on one of two hyperbolas. We show that every point in any fixed higher dimension whose coordinates are consecutive Fibonacci numbers also lies on one of two hyperbolae; moreover, these are the only lattice points on the hyperbolas. We also give some remarkable results concerning the positions of these points on the hyperbolas, as measured by appropriate angles. We present analogous results for points whose coordinates are consecutive Lucas numbers.

1. Introduction

E. Lucas [8] showed that every point (F_n, F_{n+1}) in the plane whose coordinates are consecutive Fibonacci numbers lies on one of the hyperbolas $y^2 - yx - x^2 = \pm 1$ (see Figure 1). We show that every point $(F_n, F_{n+1}, \ldots, F_{n+d-1})$ in $\mathbb{R}^d$ whose coordinates are consecutive Fibonacci numbers also lies on one of two hyperbolas; moreover, these are the only lattice points on the hyperbolas. We also give some remarkable results concerning the positions of these points on the hyperbolas, as measured by appropriate angles. We present analogous results for points $(L_n, L_{n+1}, \ldots, L_{n+d-1})$ whose coordinates are consecutive Lucas numbers.

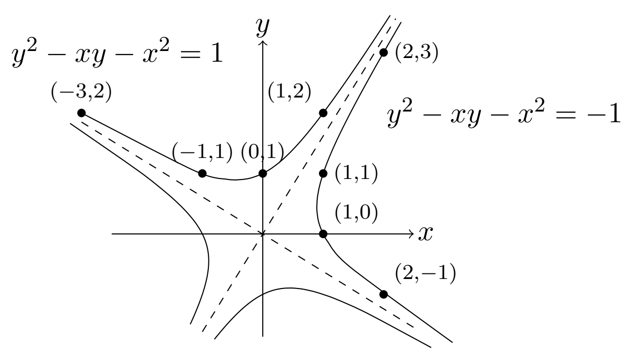

FIGURE 1. Lucas' hyperbolas in the plane

2. Visualizing Plane Curves in $\mathbb{R}^d$

Before generalizing Lucas' hyperbolas to a higher dimension, we take a moment to discuss visualizing such curves in $\mathbb{R}^d$ ($d \geq 3$).

Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a plane containing the zero vector $\vec{0}$ (if not, shift $\mathbb{R}^d$ so that it does). Fix two nonzero, nonparallel vectors $\vec{a}, \vec{b} \in \mathcal{P}$. Then $\mathcal{P}$ consists of all linear combinations of $\vec{a}$ and $\vec{b}$. Fix any unit vector $\vec{e}$ which is orthogonal to $\vec{a}$ and $\vec{b}$ such that $\vec{b}$ is to the left of $\vec{a}$ when viewing $\mathcal{P}$ from $\vec{e}$ toward $\vec{0}$ (thus specifying from which side $\mathcal{P}$ is viewed). Such an $\vec{e}$ may be constructed by fixing three coordinates such that the restrictions $\vec{a}'$ and $\vec{b}'$ of $\vec{a}$ and $\vec{b}$ to these coordinates are linearly independent, then taking $\vec{e}$ to be the unit vector which is a positive multiple of $\vec{a}' \times \vec{b}'$ on the fixed coordinates and is zero elsewhere.

Visualize curves $\mathcal{C}$ in $\mathcal{P}$ by isometrically mapping $\mathcal{P}$ to $\mathbb{R}^2$, $\vec{0}$ to $\vec{0}$, and $\vec{e}$ to the unit vector on the positive z-axis. Distances and angles of points on $\mathcal{C}$ relative to $\vec{0}$, $\vec{a}$, $\vec{b}$, and $\vec{e}$ are preserved in the image of an isometric embedding. Also, every point in a three-dimensional space is uniquely determined by its distances from four non-coplanar points. In this sense $\mathcal{C}$ viewed from $\vec{e}$ "looks identical" to its image in $\mathbb{R}^2$ viewed from above.

We describe the isometric embedding used in this paper. Let $\vec{c} := -(\vec{a}\cdot\vec{b})\vec{a} + \|\vec{a}\|^2\vec{b}$, so $\vec{b}$ and $\vec{c}$ are on the same side of $\vec{a}$ in $\mathcal{P}$. Then $\{\vec{a}/\|\vec{a}\|,\ \vec{c}/\|\vec{c}\|,\ \vec{e}\}$ is an orthonormal basis of a three-dimensional subspace of $\mathbb{R}^d$ which forms a right-hand coordinate system by construction. Let $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ be the standard basis vectors of $\mathbb{R}^3$. Let f be the linear transformation satisfying $f(\vec{a}/\|\vec{a}\|) = \hat{\imath}$, $f(\vec{c}/\|\vec{c}\|) = \hat{\jmath}$, and $f(\vec{e}) = \hat{k}$. Then f is an isometric embedding since it maps one orthonormal basis to another. We often identify points with their image under f. As such there is no need to specify $\vec{e}$. Note that $\vec{b} = \vec{a}(\vec{a}\cdot\vec{b})/\|\vec{a}\|^2 + \vec{c}/\|\vec{a}\|^2$, so $f(\vec{b}) = (\vec{a}\cdot\vec{b})/\|\vec{a}\|\hat{\imath} + \|\vec{c}\|/\|\vec{a}\|^2\hat{\jmath}$.

Suppose a curve $\mathcal{C}$ is defined parametrically by $\vec{x}(t) = u(t)\vec{a} + v(t)\vec{b}$. Then $f(\mathcal{C})$ is given by $x(t)\hat{\imath} + y(t)\hat{\jmath}$, where $x(t) = u(t)\|\vec{a}\| + v(t)(\vec{a}\cdot\vec{b})/\|\vec{a}\|$ and $y(t) = v(t)\|\vec{c}\|/\|\vec{a}\|^2$ are the x- and y-coordinate functions. The expressions for $x(t)$ and $y(t)$ may help identify $\mathcal{C}$. For example, suppose $x = x(t)$ and $y = y(t)$ satisfy $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ for all t. Then $\vec{x}$ defines a conic section in $\mathcal{P}$. Provided that A, B, C are not all zero, the type of conic section is determined by the discriminant $\Delta = B^2 - 4AC$: The curve is a hyperbola, parabola, or ellipse according to whether $\Delta > 0$, $\Delta = 0$, or $\Delta < 0$, respectively.

3. Lucas' Hyperbolas

Throughout this section all points will lie in $\mathbb{R}^d$ with $d \geq 2$. For all integers n, let the nth Fibonacci and *Lucas vectors* be the respective vectors

$$\vec{f}_n^d = \langle F_n, F_{n+1}, \ldots, F_{n+d-1}\rangle \quad \text{and} \quad \vec{\ell}_n^d = \langle L_n, L_{n+1}, \ldots, L_{n+d-1}\rangle.$$

Fibonacci and Lucas vectors were studied in [2, 9].

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Binet's formulas express the Fibonacci and Lucas numbers explicitly in terms of α and β:

$$F_n = (\alpha^n - \beta^n)/(\alpha - \beta) \quad \text{and} \quad L_n = \alpha^n + \beta^n \quad (n \in \mathbb{Z}). \tag{3.1}$$

We extend this fact using the vectors

$$\vec{a} = \langle 1,\ \alpha,\ \alpha^2,\ \ldots,\ \alpha^{d-1}\rangle \quad \text{and} \quad \vec{b} = \langle 1,\ \beta,\ \beta^2,\ \ldots,\ \beta^{d-1}\rangle.$$

The ith coordinate of $\alpha^n\vec{a} - \beta^n\vec{b}$ is $\alpha^{n+i-1} - \beta^{n+i-1}$, which by (3.1) equals $(\alpha-\beta)F_{n+i-1}$. Similarly, $\alpha^n\vec{a} - \beta^n\vec{b}$ has ith coordinate L_{n+i-1}. Thus,

$$\vec{f}_n^d = \left(\frac{\alpha^n}{\alpha-\beta}\right)\vec{a} + \left(\frac{-\beta^n}{\alpha-\beta}\right)\vec{b}, \tag{3.2}$$

$$\vec{\ell}_n^d = \alpha^n\vec{a} + \beta^n\vec{b}. \tag{3.3}$$

Observe that $\vec{a}$ and $\vec{b}$ are linearly independent since $d \geq 2$. Thus they comprise a basis for a plane (2-dimensional subspace) $\mathcal{P}$ in $\mathbb{R}^d$. Equations (3.2) and (3.3) give the coordinates of the Fibonacci and Lucas vectors relative to this basis. Rewrite (3.2) as $\vec{f}_n^d = u\vec{a} + v\vec{b}$. Observe that $\alpha\beta = -1$ and $\alpha - \beta = \sqrt{5}$, so $uv = (-1)^{n-1}/5$, depending only upon the parity of n.

Parameterizing with $u = t$ gives two curves in $\mathcal{P} \subseteq \mathbb{R}^d$: $\vec{x} = t\vec{a} + (-1)^{n-1}/(5t)\vec{b}$. Equation (3.2) implies that $\vec{f}_n^d$ lies on this curve at $t = \alpha^n/(\alpha - \beta) > 0$. Similarly, $\vec{\ell}_n^d$ lies on the curve $\vec{x} = t\vec{a} + (-1)^n/t\vec{b}$ at $t = \alpha^n$.

We visualize these curves as discussed in Section 2. In the cases under consideration, $\vec{x} = u(t)\vec{a} + v(t)\vec{b}$, where $u(t) = t$ and $v(t) = \kappa/t$ for some κ. Thus $x(t) = \lambda t + \mu/t$ and $y(t) = \nu/t$ for some scalars λ, μ, and ν. If $\mu = 0$, then $xy = \lambda\nu$, and if $\mu \neq 0$, then $y^2 - (\nu/\mu)xy + \lambda\nu^2/\mu = 0$. In both cases the discriminant is the square of some number and hence positive, so these curves are hyperbolas. After recording the above discussion, we present the equations for these hyperbolas relative to orthonormal bases.

Theorem 3.1. *Let n be an integer.*

- **(i)** *The head of $\vec{f}_n^d$, i.e., the point $(F_n, F_{n+1}, \dots F_{n+d-1})$, lies on the $t > 0$ branch of the hyperbola*

$$\vec{x} = t\vec{a} + \frac{(-1)^{n-1}}{5t}\vec{b} \quad at \quad t = \frac{\alpha^n}{\alpha - \beta}.$$

- **(ii)** *The head of $\vec{\ell}_n^d$, i.e., the point $(L_n, L_{n+1}, \dots L_{n+d-1})$, lies on the $t > 0$ branch of the hyperbola*

$$\vec{x} = t\vec{a} + \frac{(-1)^{n}}{t}\vec{b} \quad at \quad t = \alpha^n.$$

We refer to the hyperbolas of Theorem 3.1 as *Lucas' hyperbolas* for Fibonacci and Lucas vectors, respectively. There are two of each type, one for even n and one for odd n. The asymptotes of Lucas' hyperbolas are the lines through the origin parallel to $\vec{a}$ and $\vec{b}$. The formulas $y^2 - yx - x^2 = \pm 1$ for Lucas' hyperbolas in the plane (Figure 1) are expressed in terms of the standard unit vectors. However, in terms of $\vec{a}$ and $\vec{b}$ they may be expressed as in Theorem 3.1. Figure 2 shows the hyperbolas in $\mathbb{R}^3$.

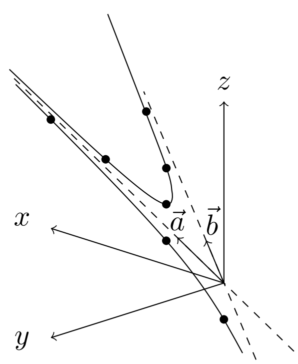

FIGURE 2. Positive branches of Lucas' hyperbolas in $\mathbb{R}^3$.

We shall now describe the hyperbolas for Fibonacci and Lucas vectors. As in Section 2, we build an orthonormal basis for $\mathcal{P}$ from $\vec{a}$ and $\vec{b}$. The following formulas were derived from

(3.1) and the sum of a finite geometric progression in [2, Lemma 2.4], [9].

$$\|\vec{a}\|^2 = \begin{cases} F_d(\alpha-\beta)\alpha^{d-1} & \text{if } d \text{ is even,} \\ L_d\alpha^{d-1} & \text{if } d \text{ is odd,} \end{cases} \tag{3.4}$$

$$\|\vec{b}\|^2 = \begin{cases} -F_d(\alpha-\beta)\beta^{d-1} & \text{if } d \text{ is even,} \\ L_d\beta^{d-1} & \text{if } d \text{ is odd,} \end{cases} \tag{3.5}$$

$$\vec{a}\cdot\vec{b} = \begin{cases} 0 & \text{if } d \text{ is even,} \\ 1 & \text{if } d \text{ is odd.} \end{cases} \tag{3.6}$$

By (3.6) the orthogonality of the asymptotes depends upon the parity of d. The asymptotes of Lucas' hyperbolas in $\mathbb{R}^2$ are known to be orthogonal [10, p. 34], and it was noted [1, p. 157] that orthogonality fails in $\mathbb{R}^3$ (Figure 2).

When d is even, $\mathcal{P}$ has orthonormal basis $\{\vec{a}/\|\vec{a}\|,\ \vec{b}/\|\vec{b}\|\}$ by (3.6). It is straightforward to express Theorem 3.1 in terms of this normalized basis.

Corollary 3.2. *Assume d is even. Let n be an integer.*

- **(i)** *The head of $\vec{f}^d_n$ lies on the hyperbola*

$$t\frac{\vec{a}}{\|\vec{a}\|} + (-1)^{n-1}\frac{F_d}{(\alpha-\beta)t}\frac{\vec{b}}{\|\vec{b}\|} \qquad at \quad t = \sqrt{\frac{F_d\alpha^{2n+d-1}}{\alpha-\beta}}.$$

- **(ii)** *The head of $\vec{\ell}^d_n$ lies on the hyperbola*

$$t\frac{\vec{a}}{\|\vec{a}\|} + (-1)^{n}\frac{F_d(\alpha-\beta)}{t}\frac{\vec{b}}{\|\vec{b}\|} \qquad at \quad t = \sqrt{F_d(\alpha-\beta)\alpha^{2n+d-1}}.$$

We visualize Corollary 3.2 in Figure 3. We identify $\vec{a}/\|\vec{a}\|$ and $\vec{b}/\|\vec{b}\|$ with $\hat{\imath}$ and $\hat{\jmath}$, as suggested in Section 2.

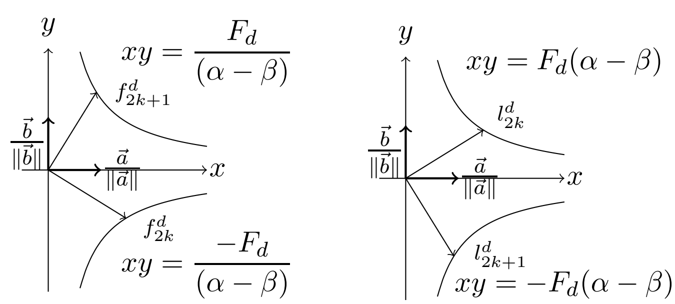

FIGURE 3. Lucas' hyperbolas when d is even.

When d is odd, $\vec{a}$ and $\vec{b}$ are not orthogonal; indeed, the cosine of the angle between $\vec{a}$ and $\vec{b}$ is $\vec{a}\cdot\vec{b}/(\|\vec{a}\|\|\vec{b}\|) = 1/L_d$. As in Section 2, define $\vec{c} = -\vec{a} + (L_d\alpha^{d-1})\vec{b}$. Equations (3.4)–(3.6) and the identity $L_d^2 - 1 = 5F_{d-1}F_{d+1}$ [3, 4] give that $\vec{a}\cdot\vec{c} = 0$ and $\|\vec{c}\|^2 = 5L_dF_{d-1}F_{d+1}\alpha^{d-1}$. Now $\{\vec{a}/\|\vec{a}\|,\ \vec{c}/\|\vec{c}\|\}$ is an orthonormal basis of $\mathcal{P}$.

Corollary 3.3. *Assume d is odd. Set $\Psi_d = (\alpha-\beta)\sqrt{F_{d-1}F_{d+1}}$. Let n be an integer.*

(i) *Let* $x(t) = \left(\frac{1}{\alpha-\beta}\right)\left(t + \frac{(-1)^{n-1}}{t}\right)$ *and* $y(t) = \frac{(-1)^{n-1}\Psi_d}{(\alpha-\beta)t}$. *Then* $x(t)$ *and* $y(t)$ *satisfy* $y(t)^2 - \Psi_d x(t)y(t) = (-1)^n \Psi_d^2/5$, *and the head of* $\vec{f}_n^d$ *lies on the hyperbola*

$$x(t)\frac{\vec{a}}{\|\vec{a}\|} + y(t)\frac{\vec{c}}{\|\vec{c}\|} \quad \text{at} \quad t = \sqrt{L_d\alpha^{2n+d-1}}.$$

(ii) *Let* $x(t) = t + \frac{(-1)^n}{t}$ *and* $y(t) = \frac{(-1)^n\Psi_d}{t}$. *Then* $x(t)$ *and* $y(t)$ *satisfy* $y(t)^2 - \Psi_d x(t)y(t) = (-1)^{n-1}\Psi_d^2$, *and the head of* $\vec{\ell}_n^d$ *lies on the hyperbola*

$$x(t)\frac{\vec{a}}{\|\vec{a}\|} + y(t)\frac{\vec{c}}{\|\vec{c}\|} \quad \text{at} \quad t = \sqrt{L_d\alpha^{2n+d-1}}.$$

Corollary 3.3 is visualized in Figure 4.

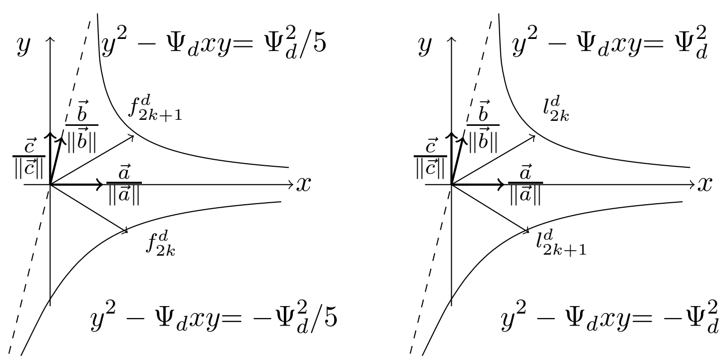

Figure 4. Lucas' hyperbolas when d is odd.

4. Lattice Points on Lucas' Hyperbolas

We generalize the following results of J. P. Jones [5, 6] to Lucas' hyperbolas.

Theorem 4.1. *Fix* $(n_1, n_2) \in \mathbb{Z}^2$.

(i) *The point* (n_1, n_2) *lies on one of the hyperbolas* $y^2 - yx - x^2 = \pm 1$ *if and only if* $(n_1, n_2) = (F_k, F_{k+1})$ *for some integer* k [5].

(ii) *The point* (n_1, n_2) *lies on one of the hyperbolas* $y^2 - yx - x^2 = \pm 5$ *if and only if* $(n_1, n_2) = (L_k, L_{k+1})$ *for some integer* k [6].

The following result extends Theorem 4.1. We identify points with their position vector.

Theorem 4.2. *Fix* $\vec{N} \in \mathbb{Z}^d$.

(i) *The head of* $\vec{N}$ *lies on one of Lucas' hyperbolas for Fibonacci vectors in* $\mathbb{R}^d$ *if and only if* $\vec{N} = \vec{f}_k^d$ *for some integer* k.

(ii) *The head of* $\vec{N}$ *lies on one of Lucas' hyperbolas for Lucas vectors in* $\mathbb{R}^d$ *if and only if* $\vec{N} = \vec{\ell}_k^d$ *for some integer* k.

Proof. We prove (i). Write $\vec{N} = \langle n_1, n_2, \ldots, n_d\rangle$. By hypothesis, $\vec{N} = t\vec{a} \pm \frac{1}{5t}\vec{b}$ for some t. Restricting to the first two coordinates gives $\langle n_1, n_2\rangle = \langle t \pm \frac{1}{5t}, \alpha t \pm \frac{\beta}{5t}\rangle$. The right-hand side parameterizes Lucas' hyperbolas $y^2 - yx - x^2 = \pm 1$ for Fibonacci numbers in two dimensions by Theorem 3.1. Hence by Theorem 4.1, $\langle n_1, n_2\rangle = \langle F_k, F_{k+1}\rangle$ for some integer k. By Theorem 3.1, this corresponds to the point on $t\vec{a} + \frac{(-1)^k}{5t}\vec{b}$ in $\mathbb{R}^2$ with $t = \alpha^k/(\alpha-\beta)$. This value in Theorem 3.1 in $\mathbb{R}^d$ gives $\vec{N} = \vec{f}_k^d$. The proof of (ii) is similar. □

We contrast Lucas' hyperbolas to "Fibonacci hyperbolas" of [7], a family of hyperbolas in $\mathbb{R}^2$, each containing an infinite number of points (F_m, F_n).

5. Angles in Even Dimension

Assume throughout this section that d is even. In Figure 5 we compare the positions of Fibonacci and Lucas vectors on Lucas' hyperbolas, as measured by appropriate angles. We show that the cosines of the angles that $\vec{f}_n^d$ and $\vec{f}_{n-t}^{d+2t}$ make with the appropriate $\vec{a}/\|\vec{a}\|$ are equal for all positive integers t.

By (3.4) and (3.6), $\vec{f}_n^d \cdot \vec{a} = F_d\alpha^{n+d-1}$ and $\vec{\ell}_n^d \cdot \vec{a} = F_d(\alpha-\beta)\alpha^{n+d-1}$ for all integers n. In [2, 9], equations (3.1) and (3.4)–(3.6) were used to show that for all integers m and n,

$$\vec{f}_m^d \cdot \vec{f}_n^d = F_d F_{m+n+d-1},\ \vec{\ell}_m^d \cdot \vec{\ell}_n^d = 5F_d F_{m+n+d-1},\ \text{and } \vec{f}_m^d \cdot \vec{\ell}_n^d = F_d L_{m+n+d-1}.$$

In particular, $\|\vec{f}_n^d\| = \sqrt{F_d F_{2n+d-1}}$ and $\|\vec{\ell}_n^d\| = \sqrt{5F_d F_{2n+d-1}}$. The formula $\vec{u}\cdot\vec{v}/\|\vec{u}\|\|\vec{v}\|$ for the cosine of the angle between $\vec{u}$ and $\vec{v}$ gives the following.

Theorem 5.1. *The following hold for all integers m and n.*

(i) *Let ψ_n^d and λ_n^d denote the respective angles between $\vec{f}_n^d$ and $\vec{a}$ and between $\vec{\ell}_n^d$ and $\vec{a}$. Write $k = 2n+d-1$. Then*

$$\cos\psi_n^d = \cos\lambda_n^d \quad = \quad \sqrt{\frac{\alpha^k}{(\alpha-\beta)F_k}} = \left(1 - \left(\frac{\beta}{\alpha}\right)^k\right)^{-1/2}. \tag{5.1}$$

(ii) *Let $\zeta_{m,n}^d$, $\eta_{m,n}^d$, and $\theta_{m,n}^d$ denote the respective angles between $\vec{f}_m^d$ and $\vec{f}_n^d$, between $\vec{\ell}_m^d$ and $\vec{\ell}_n^d$, and between $\vec{f}_m^d$ and $\vec{\ell}_n^d$. Then*

$$\cos\zeta_{m,n}^d = \cos\eta_{m,n}^d = \frac{F_{m+n+d-1}}{\sqrt{F_{2m+d-1}F_{2n+d-1}}}, \quad \cos\theta_{m,n}^d = \frac{L_{m+n+d-1}}{\sqrt{F_{2m+d-1}F_{2n+d-1}}}.$$

Taking into account the alternating nature of the hyperbolas containing the Fibonacci and Lucas vectors, Theorem 5.1 gives the following.

Corollary 5.2. *For all positive integers t and for all integers n,*

$$\psi_n^2 = (-1)^t\psi_{n-t}^{2+2t} = -\lambda_n^2 = (-1)^{t-1}\lambda_{n-t}^{2+2t}.$$

Figure 5 is a partial illustration of Corollary 5.2, with all orthonormal bases of the $\mathcal{P}$ identified. Note that the angles have been exaggerated for clarity.

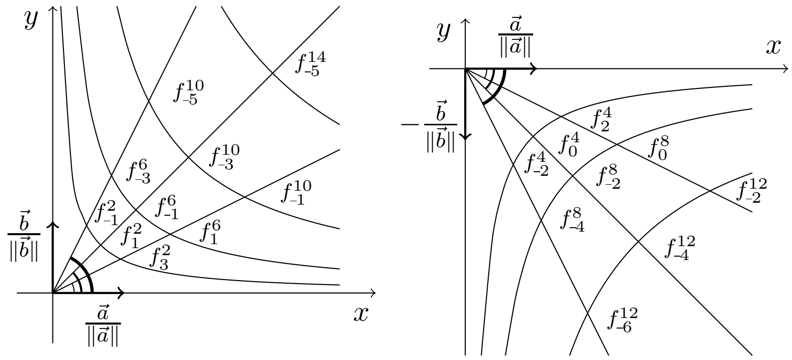

Figure 5. Angles between $\vec{a}$ and Fibonacci vectors.

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