POLYNOMIAL FORMS FOR ALTERNATING SUMS OF PRODUCTS OF BINOMIAL-CATALAN NUMBERS

N. GAUTHIER

Abstract. We study the following alternating sums,

$$f_n(m) \equiv \sum_{k\geq 0} (-1)^k \binom{n-k}{k} C_{n+m-k}, \quad n \geq 0, \quad m \geq -1$$

where C_n is the nth Catalan number, and we express the results as closed forms that can be represented as polynomials of degree m in n. We show that the number functions $f_n(m)$ are: 1) integral-valued; 2) positive-definite in sign; and 3) have a common factor, $n+1$, for $n \geq 0$, $m \geq 1$. We also show how to obtain the coefficients in the polynomial representation in powers of n.

1. Introduction

The following identity is proved in [1] and mentioned in [2]:

$$\sum_{k\geq 0} (-1)^k \binom{n-k}{k} C_{n-1-k} = \delta_{n,1}, \quad n \geq 0. \tag{1.1}$$

In this expression, n is a nonnegative integer, δ is the Kronecker delta, and C_n is the nth Catalan number [2]. The Catalan number C_{n+1} may be found recursively [2], through

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad n \geq 0, \quad C_0 = 1.$$

A Catalan number with a negative index vanishes, by definition. Likewise, a binomial coefficient with a negative upper and/or lower entry and one with a lower entry that exceeds the upper one is taken to vanish. The first few Catalan numbers are: $C_0 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 5$, $C_4 = 14$, $C_5 = 42$, $C_6 = 132$, $C_7 = 429$.

In this note, we generalize the summation identity given in (1.1) and investigate the rather singular properties of the following sums, where m and n are integers:

$$f_n(m) \equiv \sum_{k\geq 0} (-1)^k \binom{n-k}{k} C_{n+m-k}, \quad n \geq 0, \quad m \geq -1. \tag{1.2}$$

We readily conclude from (1.2) that $f_n(m)$ is integral-valued because the binomial coefficients and the Catalan numbers are integers. However, it is not yet possible to determine the overall sign of $f_n(m)$, due to the alternating nature of the sum. We note that identity (1.1) above is the special case $m = -1$ of (1.2) so we have that:

$$f_n(-1) = \delta_{n,1} \quad n \geq 0. \tag{1.3}$$

Computational data suggest that, in the cases $m = 0, 1, 2$:

$$f_n(0) = 1; f_n(1) = n + 1; f_n(2) = \frac{1}{2}(n^2 + 5n + 4), \quad n \geq 0. \tag{1.4}$$

We shall prove the validity of these expressions later but for now, we note that the following relations hold for $n \geq 0$:

$$f_{n+1}(2) - f_n(2) = n + 3 = f_{n+2}(1) \tag{1.5}$$

$$f_{n+1}(1) - f_n(1) = 1 = f_{n+2}(0) \tag{1.6}$$

$$f_{n+1}(0) - f_n(0) = 0 = f_{n+2}(-1). \tag{1.7}$$

These results and a further examination of the cases $m = 3$ and $m = 4$ led us to the following conjectures.

Conjecture I: The number functions defined in (1.2) obey the following relation:

$$f_{n+1}(m) - f_n(m) = f_{n+2}(m-1), \quad m, n \geq 0. \tag{1.8}$$

Conjecture II: The number functions defined in (1.2) may be represented as polynomials of degree m in n.

The purpose of the present note is to prove Conjecture I, in the first instance, and to give a closed-form solution for $f_n(m)$ that may be represented as a polynomial of degree m in n, in the second instance (Conjecture II). The present topic is suitable for a course in number theory and it should prove of interest to teachers, students, scientists, engineers and number enthusiasts alike. The main results and a general solution of the problem are now presented.

2. Proof of Conjectures I and II

To prove Conjecture I, first rearrange (1.8) as follows:

$$f_{n+1}(m) - f_{n+2}(m-1) = f_n(m) \quad m, n \geq 0.$$

Then use the general expressions in (1.2) and insert the results in the above relation to get:

$$\sum_{k\geq 0} (-1)^k \left[\binom{n+1-k}{k} - \binom{n+2-k}{k} \right] C_{n+1+m-k} = \sum_{k\geq 0} (-1)^k \binom{n-k}{k} C_{n+m-k}. \tag{2.1}$$

Next, note that the term with $k = 0$ in the left-hand sum vanishes and rewrite the left-hand side (LHS) as

$$\begin{aligned} \text{LHS} &= \sum_{k\geq 1} (-1)^k \left[\binom{n+1-k}{k} - \binom{n+2-k}{k} \right] C_{n+1+m-k} \\ &= \sum_{k\geq 0} (-1)^k \left[\binom{n+1-k}{k+1} - \binom{n-k}{k+1} \right] C_{n+m-k}. \end{aligned}$$

Finally, from the Pascal identity, we have

$$\binom{n+1-k}{k+1} - \binom{n-k}{k+1} = \binom{n-k}{k}$$

and the LHS of (2.1) then becomes identical to the right-hand side (RHS) of that equation. Conjecture I is therefore true for $m, n \geq 0$. We now turn to Conjecture II.

To find expressions for $f_n(m)$ as a function of n, for a fixed value of $m \geq 0$, we first shift n to k in (1.8), then change the index label from k to $k-1$ and sum over the range $1 \leq k \leq n$. The resulting sum collapses and we get, after noting that $f_0(m) = C_m$, from (1.2):

$$f_n(m) = C_m + \sum_{k=2}^{n+1} f_k(m-1), \quad m, n \geq 0. \tag{2.2}$$

We adopt the convention that the above summation vanishes for $n = 0$, regardless of the value of m. For the case $m = 0$, we put $C_0 = 1$ in (2.2) and use (1.3) to get that

$$f_n(0) = 1 + \sum_{k=2}^{n+1} \delta_{k,1} = 1, \quad n \geq 0.$$

Then, for the case $m = 1$, we put $C_1 = 1$ in (2.2) and get that

$$f_n(1) = 1 + \sum_{k=2}^{n+1} 1 = n+1, \qquad n \geq 0.$$

As a final application of (2.2), consider the case $m = 2$ and get, with $C_2 = 2$, that

$$f_n(2) = 2 + \sum_{k=2}^{n+1} (k+1) = 2 + \frac{1}{2}n(n+5) = \frac{1}{2}(n^2 + 5n + 4), \quad n \geq 0.$$

These three results prove the computational formulas given in (1.4). It would be possible to continue on in this recursive manner to obtain higher order polynomials in n, but the method quickly becomes unwieldy. Consequently we seek a direct, closed-form solution for $f_n(m)$ by proceeding as described next.

To start with, we invoke the known result [2]

$$C_k = \frac{1}{k+1}\binom{2k}{k}, \quad k \geq 0.$$

We then have, using (1.2) for the case $n = 0$, that

$$\begin{aligned}
f_0(m) &= C_m \\
&= \frac{m+1-m}{m+1}\binom{2m}{m} \\
&= \binom{2m}{m} - \frac{m}{m+1}\binom{2m}{m} \\
&= \binom{2m}{m} - \binom{2m}{m-1}.
\end{aligned}$$

Again using (1.2) to get the case $n = 1$, we have:

$$\begin{aligned} f_1(m) &= C_{m+1} \\ &= \frac{1}{m+2}\binom{2m+2}{m+1} \\ &= \frac{2m+2}{(m+2)(m+1)}\frac{(2m+1)!}{(m+1)!m!} \\ &= \frac{2}{m+2}\binom{2m+1}{m}. \end{aligned}$$

We then further transform this result to get

$$\begin{aligned} f_1(m) &= \frac{(m+2-m)}{m+2}\binom{2m+1}{m} \\ &= \binom{2m+1}{m} - \frac{m}{(m+2)}\binom{2m+1}{m} \\ &= \binom{2m+1}{m} - \binom{2m+1}{m-1}. \end{aligned}$$

As a final example, we put $n = 0$ in (1.8) and get, with the help of the previous two results and of the Pascal identity:

$$\begin{aligned} f_2(m-1) &= f_1(m) - f_0(m) \\ &= \left[\binom{2m+1}{m} - \binom{2m+1}{m-1}\right] - \left[\binom{2m}{m} - \binom{2m}{m-1}\right] \\ &= \left[\binom{2m+1}{m} - \binom{2m}{m}\right] - \left[\binom{2m+1}{m-1} - \binom{2m}{m-1}\right] \\ &= \binom{2m}{m-1} - \binom{2m}{m-2}. \end{aligned}$$

Finally we replace m by $m+1$ in this result and get the sought quantity, $f_2(m)$:

$$f_2(m) = \binom{2m+2}{m} - \binom{2m+2}{m-1}.$$

We now prove generally that

$$f_n(m) = \binom{2m+n}{m} - \binom{2m+n}{m-1}, \quad m, n \geq 0. \tag{2.3}$$

First note that (2.3) gives $f_n(0) = 1$ for all nonnegative integers n and that it gives

$$f_0(m) = \binom{2m}{m} - \binom{2m}{m-1}$$

for all nonnegative integers m. Both results agree with what we found previously, from (1.2) directly. It therefore remains to be shown that (2.3) satisfies recurrence (1.8). We have, for the left-hand side of (1.8):

$$\begin{aligned}
f_{n+1}(m) - f_n(m) &= \left[\binom{2m+n+1}{m} - \binom{2m+n+1}{m-1}\right] - \left[\binom{2m+n}{m} - \binom{2m+n}{m-1}\right] \\
&= \left[\binom{2m+n+1}{m} - \binom{2m+n}{m}\right] - \left[\binom{2m+n+1}{m-1} - \binom{2m+n}{m-1}\right] \\
&= \binom{2m+n}{m-1} - \binom{2m+n}{m-2}.
\end{aligned}$$

The Pascal identity was used to get the last line, which equals $f_{n+2}(m-1)$, and is identical to the RHS of (1.8). We conclude that (2.3) is the sought general solution of the present problem since it satisfies (1.8), plus the boundary conditions at $(m = 0, n \geq 0)$ and at $(m \geq 0, n = 0)$. Let us now consider a few specific cases of (2.3). The case $m = 0$ has already been handled. For the next two cases we have:

$$f_n(1) = \binom{n+2}{1} - \binom{n+2}{0} = n+1$$

$$f_n(2) = \binom{n+4}{2} - \binom{n+4}{1} = \frac{1}{2}(n^2 + 5n + 4).$$

Both results agree with our previous findings. For the cases $m = 3$ and $m = 4$, we have:

$$f_n(3) = \binom{n+6}{3} - \binom{n+6}{2} = \frac{1}{6}(n^3 + 12n^2 + 41n + 30)$$

$$f_n(4) = \binom{n+8}{4} - \binom{n+8}{3} = \frac{1}{24}(n^4 + 22n^3 + 167n^2 + 482n + 336).$$

The results can be obtained with relative ease and they all support the claim that $f_n(m)$ is a polynomial of degree m in n (Conjecture II). We now show that this is generally true.

We have, from (2.3), that

$$\begin{aligned}
f_n(m) &= \frac{(n+2m)!}{(n+m)!m!} - \frac{(n+2m)!}{(n+m+1)!(m-1)!} \\
&= \frac{(n+1)}{(n+m+1)} \binom{2m+n}{m}.
\end{aligned}$$

For $m = 0$ and $m = 1$, we have that $f_n(0) = 1$ and $f_n(1) = n+1$. Now, for $m \geq 2$, we simplify the binomial coefficient in this expression and get:

$$f_n(m) = \frac{(n+1)(n+2m)(n+2m-1)\cdots(n+m+2)}{m!}, \quad m \geq 2, \quad n \geq 0. \tag{2.4}$$

The denominator in the right-hand side of this expression is independent of n and the numerator contains exactly m multiplicative binomial factors in n. Consequently, $f_n(m)$ is a polynomial of degree m in n, as conjectured, and we write that

$$f_n(m) = \frac{1}{m!} \sum_{k=0}^{m} a_k(m) n^k \quad m, n \geq 0. \tag{2.5}$$

We now give expressions for the sets of coefficients, $\{a_k(m) : 0 \leq k \leq m\}$. For the special cases $m = 0$ and $m = 1$, the polynomials are given by the first two equations in (1.4). Now, for $m \geq 2$, define the following set of m numbers:

$$b_1(m) = 1; b_2(m) = 2m; b_3(m) = 2m-1; \ldots; b_m(m) = m+2.$$

These numbers come from the binomial factors in the numerator of (2.4). It then follows, from the theory of elementary symmetric functions [3], that $a_k(m)$ is the sum of the distinct products of the above numbers, taken $m-k$ at a time. We adopt the convention that $a_m(m) = 1$ for all $m \geq 2$. The $a_k(m)$ coefficients are therefore:

$$a_m(m) = 1; a_{m-1}(m) = \sum\nolimits_{1 \leq k \leq m} b_k(m); a_{m-2}(m) = \sum\nolimits_{1 \leq j < k \leq m} b_j(m) b_k(m)$$

$$a_{m-3}(m) = \sum\nolimits_{1 \leq i < j < k \leq m} b_i(m) b_j(m) b_k(m); ...; a_0(m) = \prod_{k=1}^{m} b_k(m).$$

Consider the coefficients of $m! f_n(m)$ (see (2.5)) in the following examples:

$m = 2$: $a_2(2) = 1$; $a_1(2) = b_1(2) + b_2(2) = 5$; $a_0(2) = b_1(2) b_2(2) = 4$
$m = 3$: $a_3(3) = 1$; $a_2(3) = b_1(3) + b_2(3) + b_3(3) = 12$;
$a_1(3) = b_1(3) b_2(3) + b_1(3) b_3(3) + b_2(3) b_3(3) = 41$; $a_0(3) = b_1(3) b_2(3) b_3(3) = 30$

These coefficients agree with our earlier findings.

Before closing, we note the following important properties of $f_n(m)$:

1) One can see from (2.4) that $f_n(m)$ is positive for all $m, n \geq 0$, a feature that is not obvious from definition (1.2).
2) One can see, from (1.4) for the case $m = 1$, and from (2.4) for $m \geq 2$, that every number function $f_n(m)$ shares the common factor $n + 1$ with the others, although $f_n(m)/(n + 1)$ is not necessarily integral (e.g. $f_2(3)/3 = 28/3$ and $f_3(3)/4 = 12$).
3) We recall also, from definition (1.2), that $f_n(m)$ is always integral-valued.
4) Finally, as a consequence of (2.4), each individual coefficient of a given polynomial is positive, a feature that is somewhat surprising, considering that the sum defining (1.2) is alternating in sign.

To conclude, we studied the alternating sums defined in (1.2) and expressed the results in closed form. We further showed that these sums may be represented as integral-valued, positive polynomials of degree m in the integer n and gave general expressions for the polynomial coefficients.

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Department of Physics, The Royal Military College of Canada, 17,000 Stn Forces, Kingston, Ontario, K7K 7B4, Canada
E-mail address: gauthier-n@rmc.ca