SOME CONNECTIONS BETWEEN A GENERALIZED TRIBONACCI TRIANGLE AND A GENERALIZED FIBONACCI SEQUENCE

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Abstract. In this paper we consider a generalized Fibonacci type second order linear recurrence $\{U_n\}$. We derive explicit formulas for the squares of generalized Fibonacci numbers, U_n^2, and the products of consecutive generalized Fibonacci numbers, U_nU_{n+1}, by using some properties of the generalized tribonacci triangle.

1. Introduction

For real numbers a and b, the generalized Fibonacci sequence $\{U_n\}$ is defined by

$$U_0 = 0, \quad U_1 = 1 \quad \text{and} \quad U_{n+1} = aU_n + bU_{n-1} \ (n \geq 1).$$

If $a = b = 1$, then $U_n = F_n$ is the classical Fibonacci number. It is well-known that the Fibonacci numbers can be derived by summing elements on the rising diagonal lines in Pascal's triangle

$$F_{n+1} = \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{n-i}{i} \quad (n \geq 0),$$

where $\lfloor x \rfloor$ is the largest integer not exceeding x, see [3, chapter 12]. For the generalized Fibonacci number U_n, we have the following well-known expansion, see [5],

$$U_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n-i}{i} a^{n-2i} b^i \quad (n \geq 0).$$

In 1977, Alladi and Hoggatt [1] constructed the tribonacci triangle, see Figure 1, to derive the expansion of the tribonacci numbers.

	0	1	2	3	4	5	6	7	$\cdots$
0	1								
1	1	1							
2	1	3	1						
3	1	5	5	1					
4	1	7	13	7	1				
5	1	9	25	25	9	1			
6	1	11	41	63	41	11	1		
7	1	13	61	129	129	61	13	1	
$\vdots$			$\vdots$						

Figure 1 : Tribonacci triangle.

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If we use $B(n, i)$ to denote the element in the nth row and ith column of the tribonacci triangle, then we may obtain:

$$B(n+1, i) = B(n, i) + B(n, i-1) + B(n-1, i-1), \tag{1.1}$$

where $B(n, 0) = B(n, n) = 1$. Alladi and Hoggatt showed that the sum of elements on the rising diagonal lines in the tribonacci triangle is the tribonacci number t_n, that is,

$$t_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} B(n-i, i), \tag{1.2}$$

where $t_0 = 0, t_1 = t_2 = 1$ and $t_{n+2} = t_{n+1} + t_n + t_{n-1}$.

P. Barry [2, Example 16] proved that

$$B(n, i) = \sum_{j=0}^{i} \binom{i}{j} \binom{n-j}{i}. \tag{1.3}$$

By using the identity (1.3), the identity (1.2) can be written as

$$t_{n+1} = \sum_{i=0}^{\lfloor n/2 \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{n-i-j}{i}.$$

The objective here is to find connections between a generalized tribonacci triangle and a generalized Fibonacci sequence. First, we state some formulas for the numbers F_n^2 and $F_n F_{n+1}$ suggested from the tribonacci triangle. Next, we define a generalized tribonacci triangle and derive the formulas of the numbers U_n^2 and $U_n U_{n+1}$. Their proofs will be given in the last section.

2. Skipping Rows in the Tribonacci Triangle

We delete the odd-numbered rows in the tribonacci triangle to obtain Figure 2 as follows:

	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
0	1											
2	1	3	1									
4	1	7	13	7	1							
6	1	11	41	63	41	11	1					
8	1	15	85	231	321	231	85	15	1			
10	1	19	145	575	1289	1683	1289	575	145	19	1	
12	1	23	221	1159	3649	7183	$\cdots$					
$\vdots$			$\vdots$									

Figure 2.

Observe that the sums of elements on each rising diagonal line in Figure 2 give the squared Fibonacci numbers, F_n^2, namely

$$F_1^2 = 1, F_2^2 = 1, F_3^2 = 1 + 3 = 2^2, F_4^2 = 1 + 7 + 1 = 3^2, F_5^2 = 1 + 11 + 13 = 5^2,$$

$$F_6^2 = 1 + 15 + 41 + 7 = 8^2, F_7^2 = 1 + 19 + 85 + 63 + 1 = 13^2, \ldots.$$

Using the identity (1.3), we conjecture the following expansion

$$F_{n+1}^2 = \sum_{i=0}^{\lfloor 2n/3 \rfloor} B(2n-2i, i) = \sum_{i=0}^{\lfloor 2n/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j}\binom{2n-2i-j}{i}. \tag{2.1}$$

Next, we delete the even-numbered rows in the tribonacci triangle to obtain Figure 3.

	0	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
1	1	1											
3	1	5	5	1									
5	1	9	25	25	9	1							
7	1	13	61	129	129	61	13						
9	1	17	113	377	681	681	377	113	17	1			
11	1	21	181	833	2241	3653	3653	2241	833	181	21	1	
13	1	25	265	1561	5641	13073	$\cdots$						
$\vdots$			$\vdots$										

Figure 3.

Similarly, sums of elements on each rising diagonal line in Figure 3 would appear to give the products of the consecutive Fibonacci numbers, F_nF_{n+1}, leading to the conjecture:

$$F_nF_{n+1} = \sum_{i=0}^{\lfloor (2n-1)/3 \rfloor} B(2n-2i-1, i) = \sum_{i=0}^{\lfloor (2n-1)/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j}\binom{2n-2i-j-1}{i}. \tag{2.2}$$

We will in fact prove generalized versions of (2.1) and (2.2) in the following section.

3. Main Results

Definition 3.1. *Let $n \in \mathbb{Z}$. For any non-negative integer i, let*

$$T(n,i) = \begin{cases} \sum_{j=0}^{i} \binom{i}{j}\binom{n-j}{i} a^{n-2j} b^{i+j} & ; \quad 0 \le i \le n \\ 0 & ; \quad \textit{otherwise} \end{cases}.$$

For $0 < i < n$, we see that all the terms in the summation of $T(n,i)$ are zero when $j > \min\{n-i, i\}$.

Definition 3.2. *The* ***generalized tribonacci triangle*** *is defined as follows:*

	0	1	2	3	4	5	6	$\cdots$	n	$\cdots$
0	$T(0,0)$									
1	$T(1,0)$	$T(1,1)$								
2	$T(2,0)$	$T(2,1)$	$T(2,2)$							
3	$T(3,0)$	$T(3,1)$	$T(3,2)$	$T(3,3)$						
4	$T(4,0)$	$T(4,1)$	$T(4,2)$	$T(4,3)$	$T(4,4)$					
5	$T(5,0)$	$T(5,1)$	$T(5,2)$	$T(5,3)$	$T(5,4)$	$T(5,5)$				
6	$T(6,0)$	$T(6,1)$	$T(6,2)$	$T(6,3)$	$T(6,4)$	$T(6,5)$	$T(6,6)$			
$\vdots$			$\vdots$							
n	$T(n,0)$	$T(n,1)$	$T(n,2)$			$\cdots$			$T(n,n)$	
$\vdots$			$\vdots$							

The generalized tribonacci triangle.

It is easy to see that for $a = b = 1$, the generalized tribonacci triangle is indeed the tribonacci triangle. Applying the same idea mentioned in Section 2 (i.e. deleting odd- and even-numbered rows) to the generalized tribonacci triangle, we anticipate the following main results, whose proofs will be given in the last section.

Theorem 3.3. *For any non-negative integer n, we have*

$$(1)\ U_{n+1}^2 = \sum_{i=0}^{\lfloor 2n/3 \rfloor} T(2n-2i, i) = \sum_{i=0}^{\lfloor 2n/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{2n-2i-j}{i} a^{2(n-i-j)} b^{i+j}.$$

$$(2)\ U_n U_{n+1} = \sum_{i=0}^{\lfloor (2n-1)/3 \rfloor} T(2n-2i-1, i) = \sum_{i=0}^{\lfloor (2n-1)/3 \rfloor} \sum_{j=0}^{i} \binom{i}{j} \binom{2n-2i-j-1}{i} a^{2(n-i-j)-1} b^{i+j}.$$

4. Proof of Theorem 3.3

We first provide two lemmas which will be used in the proof of Theorem 3.3.

Lemma 4.1. *Let $n \in \mathbb{N}$. Then*

(1) $U_{n+2}^2 = (a^2+b)U_{n+1}^2 + (a^2b+b^2)U_n^2 - b^3U_{n-1}^2$.
(2) $U_{n+1}U_{n+2} = (a^2+b)U_nU_{n+1} + (a^2b+b^2)U_{n-1}U_n - b^3U_{n-2}U_{n-1}$.

Proof. We only give a proof for the first part as that of the second is similar. By the definition of U_n, we get

$$\begin{aligned}
U_{n+2}^2 &= (aU_{n+1} + bU_n)^2 \\
&= a^2U_{n+1}^2 + 2abU_{n+1}U_n + b^2U_n^2 \\
&= a^2U_{n+1}^2 + abU_n(aU_n + bU_{n-1}) + bU_{n+1}(U_{n+1} - bU_{n-1}) + b^2U_n^2 \\
&= (a^2+b)U_{n+1}^2 + (a^2b+b^2)U_n^2 + b^2U_{n-1}(aU_n - U_{n+1}) \\
&= (a^2+b)U_{n+1}^2 + (a^2b+b^2)U_n^2 - b^3U_{n-1}^2,
\end{aligned}$$

as desired. □

Lemma 4.2. *Let $n \in \mathbb{N}$. For non-negative integer $i \le n$, we get that*

$$T(n,i) = aT(n-1,i) + abT(n-1,i-1) + b^2T(n-2,i-1). \tag{4.1}$$

Proof. We see that (4.1) holds for $i = 1$. For $1 < i \leq n$. We have

$$\begin{aligned}
&aT(n-1,i) + abT(n-1,i-1) + b^2T(n-2,i-1) \\
&\qquad = a\sum_{j=0}^{i}\binom{i}{j}\binom{n-j-1}{i}a^{n-2j-1}b^{i+j} + ab\sum_{j=0}^{i-1}\binom{i-1}{j}\binom{n-j-1}{i-1}a^{n-2j-1}b^{i+j-1} \\
&\qquad\quad + b^2\sum_{j=0}^{i-1}\binom{i-1}{j}\binom{n-j-2}{i-1}a^{n-2j-2}b^{i+j-1} \\
&\qquad = \binom{n-1}{i}a^nb^i + a\sum_{j=1}^{i-1}\binom{i}{j}\binom{n-j-1}{i}a^{n-2j-1}b^{i+j} + \binom{n-i-1}{i}a^{n-2i}b^{2i} \\
&\qquad\quad + \binom{n-1}{i-1}a^nb^i + ab\sum_{j=1}^{i-1}\binom{i-1}{j}\binom{n-j-1}{i-1}a^{n-2j-1}b^{i+j-1} \\
&\qquad\quad + b^2\sum_{j=0}^{i-2}\binom{i-1}{j}\binom{n-j-2}{i-1}a^{n-2j-2}b^{i+j-1} + \binom{n-i-1}{i-1}a^{n-2i}b^{2i} \\
&\qquad = \binom{n}{i}a^nb^i + \sum_{j=1}^{i-1}\binom{i}{j}\binom{n-j}{i}a^{n-2j}b^{i+j} + \binom{n-i}{i}a^{n-2i}b^{2i} \\
&\qquad = \sum_{j=0}^{i}\binom{i}{j}\binom{n-j}{i}a^{n-2j}b^{i+j} \\
&\qquad = T(n,i),
\end{aligned}$$

so (4.1) is always valid. □

Note that if we take $a = b = 1$ in the identity (1) of the Lemma 4.1, then we obtain the classical Fibonacci numbers identity, namely

$$F_{n+2}^2 = 2F_{n+1}^2 + 2F_n^2 - F_{n-1}^2,$$

which is well-known (see [4] or [3, page 92]). If we take $a = b = 1$ in Lemma 4.2, then the identity (4.1) becomes the identity (1.1).

Proof of Theorem 3.3. Since the proofs of both part (1) and part (2) are quite similar, we only give a proof for part (1). We proceed by induction on n, noting first that

$$U_1^2 = 1, \quad U_2^2 = a^2 \quad \text{and} \quad U_3^2 = a^4 + 2a^2b + b^2.$$

Now assume the identity (1) of Theorem 3.3 holds for all integers $n = 0, 1, 2, \ldots, k-1$. By Lemma 4.1(1), Lemma 4.2 and the inductive hypothesis, we get

$$
\begin{aligned}
U_{k+1}^2 &= (a^2+b)U_k^2 + (a^2b+b^2)U_{k-1}^2 - b^3U_{k-2}^2 \\
&= a^2T(2k-2,0) + a^2\sum_{i\geq 1}T(2k-2i-2,i) + b\sum_{i\geq 0}T(2k-2i-2,i) \\
&\quad + (a^2b+b^2)\sum_{i\geq 0}T(2k-2i-4,i) - b^3\sum_{i\geq 0}T(2k-2i-6,i) \\
&= T(2k,0) + a^2\sum_{i\geq 1}T(2k-2i-2,i) + bT(2k-2,0) \\
&\quad + ab\sum_{i\geq 1}T(2k-2i-3,i) + ab^2\sum_{i\geq 1}T(2k-2i-3,i-1) \\
&\quad + b^3\sum_{i\geq 1}T(2k-2i-4,i-1) - b^3\sum_{i\geq 0}T(2k-2i-6,i) \\
&\quad + (a^2b+b^2)\sum_{i\geq 1}T(2k-2i-2,i-1) \\
&= T(2k,0) + a^2\sum_{i\geq 1}T(2k-2i-2,i) + a^2b\sum_{i\geq 1}T(2k-2i-2,i-1) \\
&\quad + ab^2\sum_{i\geq 1}T(2k-2i-3,i-1) + ab\sum_{i\geq 1}T(2k-2i-1,i-1) \\
&\quad + b^2\sum_{i\geq 1}T(2k-2i-2,i-1) \\
&= T(2k,0) + a\sum_{i\geq 1}T(2k-2i-1,i) + ab\sum_{i\geq 1}T(2k-2i-1,i-1) \\
&\quad + b^2\sum_{i\geq 1}T(2k-2i-2,i-1) \\
&= \sum_{i\geq 0}T(2k-2i,i).
\end{aligned}
$$

Thus (1) of Theorem 3.3 holds for $n = k$, thereby proving the theorem.

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References

[1] K. Alladi and V. E. Hoggatt, Jr., *On tribonacci numbers and related functions*, The Fibonacci Quarterly, **15.1** (1977), 42–45.

[2] P. Barry, *On integer-sequence-based constructions of generalized Pascal triangles*, Journal of Integer Sequences, **9** (2006), Article 06.2.4.

[3] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, Wiley, New York, 2001.

[4] B. Lewis, *More power to Fibonacci*, Mathematical Gazette, **87.509** (2003), 194–202.

[5] Z. H. Sun, *Expansions and identities concerning Lucas sequences*, The Fibonacci Quarterly, **44.2** (2006), 145–153.

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