SOPHIE GERMAIN PRIMES AND THE EXCEPTIONAL VALUES OF THE EQUAL-SUM-AND-PRODUCT PROBLEM

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ABSTRACT. Using elementary methods we show that if an integer $n > 2$ is an exceptional value of the equal-sum-and-product-problem, then $n-1$ must be a Sophie Germain prime number. This result gives further evidence to the sparsity conjecture for the set of exceptional values of the equal-sum-and-product problem.

1. INTRODUCTION

The study of numbers abounds in problems which are easily understood by the non-expert, but whose solution can remain frustratingly elusive to the most knowledgeable of experts in the field. A good example of this type of problem can be found in the equal-sum-and-product problem, which so far has been examined in [1] and [2]. This problem requires, for a given integer $n \geq 2$, the determination of the set of n-tuples of positive integers $(x_1, x_2, \ldots, x_n)$ satisfying the equation

$$x_1 + x_2 + \cdots + x_{n-1} + x_n = x_1 x_2 \cdots x_{n-1} x_n. \tag{1.1}$$

For any given integer $n \geq 2$, there always exists at least one solution to the equal-sum-and-product problem, namely $(n, 2, \underbrace{1, 1 \ldots, 1}_{(n-2)1's})$, and this is referred to in [1] as the basic solution. Despite its outward simplicity, there are many difficult and unanswered questions connected with the equal-sum-and-product problem which have been identified in the references already cited. Among these is the behavior of the counting function $f(n)$, defined as the number of ordered n-tuples of positive integer solutions to (1.1). In particular, does the set of exceptional values of the equal-sum-and-product problem, defined as $E = \{n \in \mathbb{N} \backslash \{1\} : f(n) = 1\}$, have an infinite number of elements? That is, do there exist infinitely many integers $n \geq 2$ such that the basic solution is the only n-tuple of positive integers satisfying (1.1)? It is conjectured that the set of exceptional values is finite with $E = \{2, 3, 4, 6, 24, 114, 174, 444\}$ representing the complete set of exceptional values. At present, as reported in [1], extensive computer searches of all integers less than or equal to 10^{10} in specific residue classes have not revealed any new elements in E. As a consequence, it would be reasonable to expect that the set E must be sparse, regardless of whether the cardinality of E is finite or infinite. In this note we shall prove that, for an integer $n > 2$ to be an element of E, whether the cardinality of E is finite or infinite, both $n-1$ and $2n-1$ have to be prime numbers. That is, $n-1$ must, by definition, be a Sophie Germain prime number. Owing to the relative scarcity of the Sophie Germain primes compared with the sequence of ordinary prime numbers, one can see this result gives supporting evidence of the sparsity claim for the set E, and, moreover, provides additional information for a more refined computer search of possible additional elements of the set E.

2. Preliminaries and Main Result

We begin by stating and reproving, for completeness, some preliminary results that will be required later. In [1] it was shown that the equal-sum-and-product problem in two variables has exactly one solution, namely, $\{(x_1, x_2) \in \mathbb{N} \times \mathbb{N} : x_1x_2 = x_1 + x_2\} = \{(2, 2)\}$. We present here an alternate proof of this fact based on a simple divisibility argument, which will later form the basis for the proof of the main result. Suppose $x_1x_2 = x_1 + x_2$, for some positive integers x_1, x_2. Then as $x_1|(x_1 + x_2)$, observe $x_1|x_2$ and so $x_1 \leq x_2$. Similarly $x_2 \leq x_1$. Consequently, $x_1 = x_2$ and so $x_1^2 = 2x_1$ which yields that either $x_1 = 2$ or $x_1 = 0$, but as x_1 is a positive integer we conclude $x_1 = x_2 = 2$. Next, in [1] it was also shown, via an inductive argument, that apart from the two-variable case, the equal-sum-and-product problem in $n \geq 3$ variables cannot have solutions $(x_1, x_2, \ldots, x_n)$ in which $x_i \geq 2$ for all $1 \leq i \leq n$. This can be proved directly as follows. Assuming $x_i \geq 2$ for all $1 \leq i \leq n$ and after reordering and relabeling letting $x_i \geq x_{i+1}$, observe that $x_1x_2 \cdots x_n \geq 2^{n-1}x_1$ while $nx_1 > x_1 + x_2 + \cdots + x_n$, but as $2^{n-1} > n$ for $n \geq 3$ we deduce that $x_1x_2 \cdots x_n > x_1 + x_2 + \cdots + x_n$, and so cannot be a solution of (1.1). In view of these results we introduce, for notational convenience, the following definition for classifying equal-sum-and-product solutions, according to the number of non-unit components present within the n-tuple.

Definition 2.1. *For given integers $n \geq r \geq 2$, let $(x_1, x_2, \ldots, x_r; n - r)$ denote an n-tuple having r non-unit components and satisfying equation (1.1), and set*

$$S_r(n) = \left\{ (x_1, x_2, \ldots, x_r; n - r) \in \mathbb{N}^n : \prod_{j=1}^{r} x_j = \sum_{j=1}^{r} x_j + n - r \right\}.$$

In view of Definition 2.1, the basic solution for the equal-sum-and-product problem is denoted as $(n, 2; n - 2)$. Now for an integer $n > 2$ to be an element of E, then necessarily $S_2(n) = \{(n, 2; n - 2)\}$ and $S_3(n) = \emptyset$. By applying this simple implication we can now prove our main result.

Theorem 2.2. *If an integer $n > 2$ is an exceptional value of the equal-sum-and-product problem, then $n - 1$ must be a Sophie Germain prime number.*

Proof. We note by inspection that the known elements n of $E\backslash\{2\}$ satisfy the property that $n - 1$ is a Sophie Germain prime number, and so in what follows we shall assume there exists another element of E with $n > 444$. Our first task will be to characterize the set $S_2(n)$ in terms of the divisors of $n - 1$. Suppose $(x_1, x_2; n - 2) \in S_2(n)$. Then upon rearrangement of $x_1 + x_2 + n - 2 = x_1x_2$, we find $n - 1 = (x_1 - 1)(x_2 - 1)$. Assuming $x_1 \geq x_2$, observe that for a divisor d of $n - 1$ with $d \leq \sqrt{n - 1}$, we may set $x_2 - 1 = d$ and $x_1 - 1 = (n - 1)/d$ and so

$$S_2(n) = \left\{ \left(\frac{n-1}{d} + 1, d + 1; n - 2 \right) : d|(n - 1), d \leq \sqrt{n - 1} \right\}.$$

Consequently, the cardinality of $S_2(n)$ is equal to the number of ordered integer factorizations of $n - 1 = ab$, with $a \geq b \geq 1$. Thus, $S_2(n) = \{(n, 2; n - 2)\}$ if and only if $n - 1$ is a prime number. Next, we further establish a characterization for the set $S_3(n)$ in terms of the preceding sets $S_2(\cdot)$ as follows:

$$S_3(n) = \bigcup_{j+1=0}^{\lfloor \frac{n-5}{2} \rfloor} \left\{ (x_1, x_2, x_3; n - 3) : (x_1, x_2; j + 1) \in S_2(3 + j), x_3 := \frac{x_1 + x_2 + n - 3}{x_1x_2 - 1} \in \mathbb{N} \right\}. \tag{2.1}$$

Suppose there exists integers $x_1, x_2, x_3 \geq 2$ such that $x_1x_2x_3 = x_1 + x_2 + x_3 + n - 3$. Solving for x_3 we find that

$$x_3 = \frac{x_1 + x_2 + n - 3}{x_1x_2 - 1} \in \mathbb{N}, \tag{2.2}$$

and so $(x_1x_2 - 1)|(x_1 + x_2 + n - 3)$. Now as $x_3 \geq 2$, we can conclude that $(x_1x_2 - 1) < x_1 + x_2 + n - 3$ and so there must exist a $k \in \mathbb{N}$ such that $x_1x_2 - 1 = (x_1 + x_2 + n - 3) - k$, or equivalently after setting $j = n - 3 - k$

$$x_1x_2 = x_1 + x_2 + (j + 1), \tag{2.3}$$

with $j + 1 \geq 0$, as $x_1, x_2 \geq 2$. Thus $(x_1, x_2; j + 1) \in S_2(3 + j)$. Conversely, suppose $(x_1, x_2; j + 1) \in S_2(3 + j)$ with $3 + j < n$ and such that $x_3 := \frac{x_1+x_2+n-3}{x_1x_2-1} \in \mathbb{N}$. Then by definition of x_3 we have $(x_1, x_2, x_3; n - 3) \in S_3(n)$, noting here that $x_3 \geq 2$, as $x_3 \neq 1$, since by definition as $x_1x_2 = x_1 + x_2 + j + 1$ we have

$$\begin{aligned}\frac{x_1 + x_2 + n - 3}{x_1x_2 - 1} &= \frac{x_1 + x_2 + n - 3}{x_1 + x_2 + j} \\ &= \frac{(x_1 + x_2 + j) + n - 3 - j}{x_1 + x_2 + j} \\ &= 1 + \frac{n - 3 - j}{x_1 + x_2 + j} \\ &= 1 + \frac{k}{x_1 + x_2 + j} > 1.\end{aligned}$$

Thus to generate $S_3(n)$, it suffices to find solutions $(x_1, x_2; j + 1) \in S_2(3 + j)$, satisfying the divisibility condition in (2.2), and construct $S_3(n)$ as the set containing elements of the form $(x_1, x_2, x_3; n - 3)$, with x_3 as defined above. We now give an upper bound for the term $j + 1$. Recalling for $(x_1, x_2; j + 1) \in S_2(3 + j)$ that $x_1x_2 = x_1 + x_2 + j + 1$, observe again from above

$$\frac{x_1 + x_2 + n - 3}{x_1x_2 - 1} = 1 + \frac{n - 3 - j}{x_1 + x_2 + j}. \tag{2.4}$$

Now the right-hand side of (2.4) will not be an integer if $n - 3 - j < x_1 + x_2 + j$. Furthermore as $x_1, x_2 \geq 2$, observe that the previous inequality will be satisfied if and only if $n - 3 - j < 2 + 2 + j$, namely when $j + 1 > \frac{n-5}{2}$. Consequently we need only examine those sets $S_2(3 + j)$, where $0 \leq j + 1 \leq \lfloor \frac{n-5}{2} \rfloor$. Now if an integer $n > 2$ is an element of E, then also $S_3(n) = \emptyset$, and so from (2.1) we deduce upon substituting the basic solution $(3 + j, 2; j + 1) \in S_2(j + 3)$ into the expression for x_3, that $x_3 := \frac{j+2+n}{2j+5} \notin \mathbb{N}$, for $j + 1 = 0, \ldots, \lfloor \frac{n-5}{2} \rfloor$. As $2j + 5$ is an odd integer the previous conclusion further implies that $2x_3 \notin \mathbb{N}$, and so observe

$$2x_3 := \frac{2j + 4 + 2n}{2j + 5} \tag{2.5}$$

$$= \frac{2j + 5 + (2n - 1)}{2j + 5}$$

$$= 1 + \frac{2n - 1}{2j + 5} \notin \mathbb{N}, \tag{2.6}$$

for $j + 1 = 0, \ldots, \lfloor \frac{n-5}{2} \rfloor$. Recalling that $n - 1$ is prime and so n is an even integer, one finds that $\lfloor \frac{n-5}{2} \rfloor = \lfloor \frac{n-4}{2} - \frac{1}{2} \rfloor = \frac{n-4}{2} - 1$, consequently $2j + 5$ assumes the values of all odd integers $3, 5, \ldots, n - 3 \leq \sqrt{2n - 1} < n$, for $j + 1 = 0, \ldots, \lfloor \frac{n-5}{2} \rfloor$. As $n - 1$ clearly does not divide

$2n-1 = 2(n-1)+1$, we deduce from (2.6) that $2n-1$ must be a prime number and so by definition $n-1$ is a Sophie Germain prime number. □

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References

[1] M. W. Ecker, *When does a sum of positive integers equal their product?*, Mathematics Magazine, **75.1** (2002), 41–47.

[2] L. Kurlandchik, *When the sum equals the product*, The Mathematical Gazette, **84.499** (2000), 91–94.

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