SOME INTERESTING INFINITE FAMILIES OF PRIMITIVE PYTHAGOREAN TRIPLES

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Abstract. In this paper we investigate families of primitive Pythagorean triples of the form (a,b,c), where $mc - nb = t$, $mc - na = t$, or $mb - na = t$ for some fixed positive coprime integers m and n, and t a fixed nonzero integer. A few of these cases are especially interesting since the solutions may be simply written in terms of Fibonacci and Lucas numbers.

1. Introduction

Pythagorean triples have fascinated mathematicians for over 3000 years. Due to the celebrated Pythagorean Theorem, Pythagorean triples correspond to right triangles with integer sides. In this paper we investigate some very interesting infinite families of primitive Pythagorean triples, in which the ratio of the lengths of two specified sides approaches an arbitrary fixed rational value m/n, where m and n are positive coprime integers. A few of these cases are particularly interesting since the solutions may be simply written in terms of Fibonacci and Lucas numbers.

Definition 1.1. *A Pythagorean triple (PT) is a triple of positive integers (a,b,c) in which $a^2 + b^2 = c^2$.*

Definition 1.2. *A primitive Pythagorean triple (PPT) is a PT (a,b,c) in which $gcd(a,b,c) = 1$.*

Definition 1.3. *A well-ordered PPT is a PPT (a,b,c) in which a is odd and b is even.*

It is well-known that for every PPT (a,b,c), a and b are of opposite parity. Thus, every PPT may be made well-ordered simply by transposing a and b if necessary. From now on in this paper, by PPT we will always mean a well-ordered PPT.

We use Π to denote the set of all (well-ordered) PPT's. The mathematical definition of Π is as follows:

$$\Pi = \{(a,b,c) \in \mathbb{Z}^{+3} : a^2 + b^2 = c^2, 2 \nmid a, 2|b, \gcd(a,b,c) = 1\}. \tag{1.1}$$

The following theorem, due to Euclid [2], gives a very useful parametrization of all PPT's.

Theorem 1.4. *There are infinitely many PPT's and they may all be found by the following parametrization: Let u and v be positive coprime integers of opposite parity with $u > v$. Define the integers a, b, and c as follows:*

$$\begin{aligned} a &= u^2 - v^2; \\ b &= 2uv; \\ c &= u^2 + v^2. \end{aligned} \tag{1.2}$$

Then (a,b,c) is a PPT.

We omit the proof of this theorem because it can be found in numerous books, for instance, the one cited in [3].

We will also find another type of parametrization of PPT's useful, as illustrated by the following theorem.

Theorem 1.5. *All PPT's may be found by the following parametrization: Let u' and v' be odd positive coprime integers with $u' > v'$. Define the integers a, b, and c as follows:*

$$\begin{aligned} a &= u'v'; \\ b &= \tfrac{1}{2}(u'^2 - v'^2); \\ c &= \tfrac{1}{2}(u'^2 + v'^2). \end{aligned} \tag{1.3}$$

Then (a, b, c) is a PPT.

Once again we omit the proof of this theorem, but merely note that the following transformation takes us from parametrization (1.2) to parametrization (1.3):

$$\begin{aligned} u' &= u + v; \\ v' &= u - v. \end{aligned} \tag{1.4}$$

The following table lists the first few PPT's along with each type of parametrization discussed above.

u	v	u'	v'	a	b	c
2	1	3	1	3	4	5
3	2	5	1	5	12	13
4	1	5	3	15	8	17
4	3	7	1	7	24	25
5	2	7	3	21	20	29
5	4	9	1	9	40	41

Table 1. The first few PPT's and their parametrizations.

2. Definitions and Examples

In this section we make some useful definitions regarding families of PPT's and illustrate with a few examples.

Definition 2.1. *Let m and n be fixed positive coprime integers with $m \le n$ and let t be an arbitrary fixed integer. The type-1a primitive Pythagorean triple family (PPTF) $\Pi(*, m, n|t)$ is the set of PPT's (a, b, c) (with a odd and b even) such that $mc - nb = t$.*

Definition 2.2. *Let m and n be fixed positive coprime integers with $m \le n$ and let t be an arbitrary fixed integer. The type-1b PPTF $\Pi(m, *, n|t)$ is the set of PPT's (a, b, c) (with a odd and b even) such that $mc - na = t$.*

Definition 2.3. *Let m and n be fixed positive coprime integers and let t be an arbitrary fixed integer. The type-2 PPTF $\Pi(m, n, *|t)$ is the set of PPT's (a, b, c) (with a odd and b even) such that $mb - na = t$.*

Written completely in mathematical language, we have the following definitions:

$$\Pi(*, m, n|t) = \{(a, b, c) \in \Pi : mc - nb = t\}; \tag{2.1}$$

$$\Pi(m, *, n|t) = \{(a, b, c) \in \Pi : mc - na = t\}; \tag{2.2}$$

$$\Pi(m, n, *|t) = \{(a, b, c) \in \Pi : mb - na = t\}. \tag{2.3}$$

The following two definitions are also useful.

Definition 2.4. *Let m, n, and t be as before. The category-1 type-1a primitive Pythagorean triple family parametrization (PPTFP) $P(*, m, n|t)$ is the set of pairs (u, v) such that a, b, and c satisfy equations (1.2) for every PPT (a, b, c) in $P(*, m, n|t)$. Category-1 type-1b and type-2 PPTFP's are defined similarly in terms of type-1b and type-2 PPTF's, respectively.*

Definition 2.5. *Let m, n, and t be as before. The category-2 type-1a primitive Pythagorean triple family parametrization (PPTFP) $P'(*, m, n|t)$ is the set of pairs (u', v') such that a, b, and c satisfy equations (1.3) for every PPT (a, b, c) in $P(*, m, n|t)$. Category-2 type-1b and type-2 PPTFP's are defined similarly in terms of type-1b and type-2 PPTF's, respectively.*

We will now look at some special cases of PPTF's of each type.

2.1. **Type-1a; $m = n = 1$.** Here we investigate PPTF's of the form $\Pi(*, 1, 1|t)$. The following theorem classifies them all.

Theorem 2.6. *The PPTF $\Pi(*, 1, 1|t)$ is empty unless t is an odd square ($t = (2l + 1)^2$ for some nonnegative integer l), in which case we have*

$$\begin{aligned}\Pi(*, 1, 1|(2l + 1)^2) = \{&(2k(2l + 1) + (2l + 1)^2, 2k^2 + 2k(2l + 1),\\ &2k^2 + 2k(2l + 1) + (2l + 1)^2) : k \in \mathbb{Z}^+, gcd(k, 2l + 1) = 1\}.\end{aligned} \tag{2.4}$$

Proof. First we show that $\Pi(*, 1, 1|t)$ is empty unless t is an odd square. We know that (a, b, c) belongs to $\Pi(*, 1, 1|t)$ if and only if $c - b = t$. From Euclid's parametrization (1.2), this is true if and only if $u^2 + v^2 - 2uv = (u - v)^2 = t$. But since u and v have opposite parity, this is equivalent to t being an odd square.

Next we must verify equation (2.4). It is straightforward to show that every triple (a, b, c) of the form given by (2.4) satisfies $a^2 + b^2 = c^2$. It is also straightforward to show that (a, b, c) is primitive if and only if $\gcd(k, 2l+1) = 1$. Thus we see that every element of the set T on the right side of (2.4) must belong to $S = \Pi(*, 1, 1|(2l+1)^2)$, so $T \subset S$. Now consider an arbitrary element (a, b, c) of S. We know that $c - b = (u - v)^2 = (2l + 1)^2$, hence, $u - v = 2l + 1$ since $u > v$. Thus, without loss of generality we let $v = k$ and $u = k + 2l + 1$, where k is a positive integer. From (1.2), we see that $a = u^2 - v^2 = 2k(2l+1)+(2l+1)^2$, $b = 2uv = 2k^2+2k(2l+1)$, and $c = 2k^2 + 2k(2l + 1) + (2l + 1)^2$. Thus we have $S \subset T$, hence, $S = T$, verifying (2.4). $\square$

The first few examples of $\Pi(*, 1, 1|t)$ are as follows:

$$\begin{aligned}\Pi(*, 1, 1|1) &= \{(2k + 1, 2k^2 + 2k, 2k^2 + 2k + 1) : k \in \mathbb{Z}^+\}\\ &= \{(3, 4, 5), (5, 12, 13), (7, 24, 25), (9, 40, 41), \dots\};\end{aligned} \tag{2.5}$$

$$\begin{aligned}\Pi(*, 1, 1|9) &= \{(6k + 9, 2k^2 + 6k, 2k^2 + 6k + 9) : k \in \mathbb{Z}^+, 3 \nmid k\}\\ &= \{(15, 8, 17), (21, 20, 29), (33, 56, 65), (39, 80, 89), \dots\};\end{aligned} \tag{2.6}$$

$$\begin{aligned}\Pi(*, 1, 1|25) &= \{(10k + 25, 2k^2 + 10k, 2k^2 + 10k + 25) : k \in \mathbb{Z}^+, 5 \nmid k\}\\ &= \{(35, 12, 37), (45, 28, 53), (55, 48, 73), (65, 72, 97), \dots\};\end{aligned} \tag{2.7}$$

2.2. **Type-1b; $m = n = 1$.** Here we investigate PPTF's of the form $\Pi(1,*,1|t)$. The following theorem classifies them all.

Theorem 2.7. *The PPTF $\Pi(1,*,1|t)$ is empty unless t is twice a square, in which case we have*

$$\Pi(1,*,1|2(2l)^2) = \{((2k+1)^2-(2l)^2, 2(2k+1)(2l), (2k+1)^2+(2l)^2) : k \in \mathbb{Z}, k \geq l, gcd(2k+1,l)=1\}; \tag{2.8}$$

$$\Pi(1,*,1|2(2l+1)^2) = \{(2k)^2-(2l+1)^2, 2(2k)(2l+1), (2k)^2+(2l+1)^2) : k \in \mathbb{Z}, k > l, gcd(k,2l+1)=1\}. \tag{2.9}$$

Proof. First consider $S = \Pi(1,*,1|2(2l)^2)$. We have $(a,b,c) \in S$ if and only if $c-a = 2(2l)^2$. Using the parametrization in (1.2) and simplifying, we find $v = 2l$. Since v is even, u must be odd, so we may write $u = 2k+1$ for $k \geq l$. Equation (2.8) now follows easily from the formulas in (1.2) for a, b, and c. As before, the gcd condition arises from the requirement of primitivity. The derivation of (2.9) is similar. □

The first few examples of $\Pi(1,*,1|t)$ are as follows.

$$\begin{aligned}\Pi(1,*,1|2) &= \{(4k^2-1, 4k, 4k^2+1) : k \in \mathbb{Z}^+\} \\ &= \{(3,4,5),(15,8,17),(35,12,37),(63,16,65),\ldots\};\end{aligned} \tag{2.10}$$

$$\begin{aligned}\Pi(1,*,1|8) &= \{(4k^2+4k-3, 8k+4, 4k^2+4k+5) : k \in \mathbb{Z}^+\} \\ &= \{(5,12,13),(21,20,29),(45,28,53),(77,36,85),\ldots\};\end{aligned} \tag{2.11}$$

$$\begin{aligned}\Pi(1,*,1|18) &= \{(4k^2-9, 12k, 4k^2+9) : k \in \mathbb{Z}, k \geq 2, 3 \nmid k\} \\ &= \{(7,24,25),(55,48,73),(91,60,109),(187,84,205),\ldots\};\end{aligned} \tag{2.12}$$

$$\begin{aligned}\Pi(1,*,1|32) &= \{(4k^2+4k-15, 16k+8, 4k^2+4k+17) : k \in \mathbb{Z}, k \geq 2\} \\ &= \{(9,40,41),(33,56,65),(65,72,97),(105,88,137),\ldots\};\end{aligned} \tag{2.13}$$

2.3. **Type-2; $m = n = 1, t = \pm 1$.** Here we investigate the PPTF's $\Pi(1,1,*|1)$ and $\Pi(1,1,*|-1)$. Unlike in the previous two cases (which are special), each of these PPTF's corresponds to an infinite family of right triangles which are asymptotically similar to a particular one, in this case an isosceles right triangle. The following theorem classifies each of these PPTF's.

Theorem 2.8. *The PPTF's $\Pi(1,1,*|1)$ and $\Pi(1,1,*|{-1})$ are given by the following formulas:*

$$\Pi(1,1,*|1) = \{(\tfrac{1}{2}(H_{4k-1}-1), \tfrac{1}{2}(H_{4k-1}+1), P_{4k-1}), k \in \mathbb{Z}^+\}; \tag{2.14}$$

$$\Pi(1,1,*|{-1}) = \{(\tfrac{1}{2}(H_{4k+1}+1), \tfrac{1}{2}(H_{4k+1}-1), P_{4k+1}), k \in \mathbb{Z}^+\}, \tag{2.15}$$

where P_k and H_k are the kth Pell numbers and half companion Pell numbers, respectively, [8].

Proof. First we note that the triple (a,b,c) belongs to $\Pi(1,1,*|\pm 1)$ if and only if $a-b = \pm 1$. From the parametrization in (1.2), this is equivalent to

$$u^2 - 2uv - v^2 = \pm 1. \tag{2.16}$$

Letting $w = u - v$, this becomes

$$w^2 - 2v^2 = \pm 1. \tag{2.17}$$

This is the well-known Pell equation and negated Pell equation for $n = 2$, [7]. Its positive solutions are $v = P_k$ and $w = H_k$, the right side being $(-1)^k$. The numbers P_k are known as Pell numbers and the numbers H_k are called half companion Pell numbers. They satisfy the recurrences $P_{k+1} = 2P_k + P_{k-1}$ and $H_{k+1} = 2H_k + H_{k-1}$ and the initial conditions $P_0 = 0$, $P_1 = 1$ and $H_0 = H_1 = 1$. Alternatively, these numbers may be defined (analogously to the Binet formulas for F_k and L_k) in terms of $\gamma = 1 + \sqrt{2}$ and $\delta = 1 - \sqrt{2}$ as follows:

$$P_k = \frac{\gamma^k - \delta^k}{\sqrt{8}}; \tag{2.18}$$

$$H_k = \frac{\gamma^k + \delta^k}{2}. \tag{2.19}$$

k	P_k	H_k
0	0	1
1	1	1
2	2	3
3	5	7
4	12	17
5	29	41
6	70	99
7	169	239
8	408	577

Table 2. The first few Pell numbers and half companion Pell numbers.

Below we list some useful identities involving Pell numbers and half companion Pell numbers. These may all be easily proven by use of equations (2.18) and (2.19), noting that $\gamma\delta = -1$.

$$P_k + H_k = P_{k+1}; \tag{2.20}$$

$$P_{k+1}^2 - P_k^2 = \tfrac{1}{2}[H_{2k+1} + (-1)^k]; \tag{2.21}$$

$$2P_k P_{k+1} = \tfrac{1}{2}[H_{2k+1} + (-1)^{k+1}]; \tag{2.22}$$

$$P_{k+1}^2 + P_k^2 = P_{2k+1}. \tag{2.23}$$

Note that $u_k = v_k + w_k = P_k + H_k = P_{k+1}$, $v_k = P_k$, and $t_k = (-1)^k$, hence

$$P(1, 1, *|1) = \{(P_{2k}, P_{2k-1}) : k \in \mathbb{Z}^+\} \tag{2.24}$$

and

$$P(1, 1, *|-1) = \{(P_{2k+1}, P_{2k}) : k \in \mathbb{Z}^+\}. \tag{2.25}$$

Now consider the kth element (a_k, b_k, c_k) of $\Pi(1, 1, *|1)$. We have $a_k = u_k^2 - v_k^2 = P_{2k}^2 - P_{2k-1}^2 = \frac{1}{2}(H_{4k-1} - 1)$ by (2.21). Similarly we have $b_k = 2u_k v_k = 2P_{2k}P_{2k-1} = \frac{1}{2}(H_{4k-1} + 1)$ by (2.22). Finally we have $c_k = u_k^2 + v_k^2 = P_{2k}^2 + P_{2k-1}^2 = P_{4k-1}$ by (2.23). Thus we have verified (2.14). The derivation of (2.15) is similar. (For a similar derivation of this theorem, see [1].) □

Below we explicitly list the first few elements of $\Pi(1,1,*|1)$ and $\Pi(1,1,*|-1)$. Note that the coefficients grow exponentially, unlike those of the previous special examples, which grow quadratically and linearly. Exponential growth is typical for PPTF's.

$$\begin{aligned} \Pi(1,1,*|1) &= \{(3,4,5),(119,120,169),(4059,4060,5741),\dots\}; \\ \Pi(1,1,*|-1) &= \{(21,20,29),(697,696,985),(23661,23660,33461),\dots\}. \end{aligned} \tag{2.26}$$

3. Connection with Fibonacci and Lucas Numbers

We have found a fascinating connection of certain PPTF's with Fibonacci and Lucas numbers. In this section we present eight PPTF's whose coefficients may all be simply expressed in terms of Fibonacci or Lucas numbers.

Before we continue, we will find the following two theorems useful. As always, we use α to represent the golden mean, i.e. $\alpha = \frac{1}{2}(1+\sqrt{5})$.

Theorem 3.1. *Let u and v be nonnegative integers. Then u and v satisfy the equation*

$$u^2 - 5v^2 = \pm 4$$

if and only if there exists a nonnegative integer j such that $u = L_j$ and $v = F_j$, in which case we have

$$u^2 - 5v^2 = 4(-1)^j. \tag{3.1}$$

Proof. We first note that

$$u^2 - 5v^2 = (u + v\sqrt{5})(u - v\sqrt{5}) = N(u + v\sqrt{5}), \tag{3.2}$$

where $N(\gamma)$ is the norm of an element γ of the number field $K = \mathbb{Q}(\sqrt{5})$. Now since the ring of integers $R = \mathcal{O}_K = \mathbb{Z}[\alpha]$ is a UFD [5], we have

$$N(\gamma) = \pm 4$$

if and only if γ is an associate of 2, i.e if and only if $\gamma = \pm 2\alpha^j$ for some integer j, since α is the fundamental unit of K, [6]. Thus we see that $u^2 - 5v^2 = \pm 4$ if and only if $u + v\sqrt{5} = \pm 2\alpha^j$ for some integer j. Now from the Binet formulas, it is trivial to show that

$$\alpha^j = L_j + F_j\sqrt{5}. \tag{3.3}$$

Thus we have

$$\begin{aligned} u^2 - 5v^2 = \pm 4 &\Leftrightarrow u + v\sqrt{5} = \pm 2\alpha^j = \pm(L_j + F_j\sqrt{5}) \\ &\Leftrightarrow u = \pm L_j,\ v = \pm F_j \end{aligned} \tag{3.4}$$

where the signs of u_j and v_j are the same.

Note that in order for u and v to both be nonnegative, we must have $uv = F_jL_j = F_{2j} \geq 0$, which is true if and only if j is nonnegative, in which case both F_j and L_j are nonnegative, hence the $\pm$ signs on the right side of (3.4) become plus signs.

To complete the proof, we must verify the sign in equation (3.1). This follows from the fact that $N(\alpha) = -1$. □

We also have the following closely-related theorem.

Theorem 3.2. *Let u and v be nonnegative integers. Then u and v satisfy the equation*

$$u^2 - 5v^2 = \pm 1$$

if and only if there exists a nonnegative integer j such that $u = \frac{1}{2}L_{3j}$ and $v = \frac{1}{2}F_{3j}$, in which case we have

$$u^2 - 5v^2 = (-1)^j. \tag{3.5}$$

Proof. Let $U = 2u$ and $V = 2v$. Then U and V satisfy the equation

$$U^2 - 5V^2 = 4(-1)^j$$

if and only if u and v satisfy equation (3.5). If and only if this is the case, by Theorem 3.1, we have $U = L_i$ and $V = F_i$ for some nonnegative integer i. Now U and V are both even precisely when i is a multiple of 3, i.e. $i = 3j$ for some nonnegative integer j. The result follows. $\square$

We now look at various types of PPTF's whose coefficients involve Fibonacci and Lucas numbers.

3.1. **Type-1a; $m = 2, n = 3$.** Here we investigate PPTF's of the form $\Pi(*, 2, 3|t)$ for some small integer t. These PPTF's correspond to families of right triangles asymptotically similar to one with legs of length $\sqrt{5}$ and 2 and hypotenuse of length 3. The reason we might expect a connection with Fibonacci and/or Lucas numbers is that these numbers are intimately connected with the golden mean α, which lies in the number field $\mathbb{Q}(\sqrt{5})$.

By definition, a PPT (a, b, c) belongs to $\Pi(*, 2, 3|t)$ if and only if $2c - 3b = t$, which holds if and only if

$$2(u'^2 + v'^2) - 3(u'^2 - v'^2) = 2t,$$

where u' and v' are the parameters in Theorem 1.5. Thus we have

$$u'^2 - 5v'^2 = -2t.$$

The simplest nontrivial case to consider is $t = \pm 2$, in which case we have

$$u'^2 - 5v'^2 = \mp 4.$$

The following theorem classifies all PPTF's of the form $\Pi(*, 2, 3|\pm 2)$.

Theorem 3.3. *The PPTF's $\Pi(*, 2, 3|2)$ and $\Pi(2, 3, *|{-2})$ are given by the following formulas:*

$$\begin{aligned} \Pi(*, 2, 3|2) &= \{(F_{12k-2}, \tfrac{2}{5}(L_{12k-2} - 3), \tfrac{1}{5}(3L_{12k-2} - 4)), k \in \mathbb{Z}^+\} \\ &\cup \{(F_{12k+2}, \tfrac{2}{5}(L_{12k+2} - 3), \tfrac{1}{5}(3L_{12k+2} - 4)), k \in \mathbb{Z}^+\}; \end{aligned} \tag{3.6}$$

$$\begin{aligned} \Pi(*, 2, 3|{-2}) &= \{(F_{12k+4}, \tfrac{2}{5}(L_{12k+4} + 3), \tfrac{1}{5}(3L_{12k+4} + 4)), k \in \mathbb{N}\} \\ &\cup \{(F_{12k+8}, \tfrac{2}{5}(L_{12k+8} + 3), \tfrac{1}{5}(3L_{12k+8} + 4)), k \in \mathbb{N}\}. \end{aligned} \tag{3.7}$$

Proof. We start by proving the following lemma, from which Theorem 3.3 will easily follow.

Lemma 3.4. *We have*

$$P'(*, 2, 3|2) = \{(L_{6k-1}, F_{6k-1}) : k \in \mathbb{Z}^+\} \cup \{(L_{6k+1}, F_{6k+1}) : k \in \mathbb{Z}^+\} \tag{3.8}$$

and

$$P'(*, 2, 3|{-2}) = \{(L_{6k+2}, F_{6k+2}) : k \in \mathbb{N}\} \cup \{(L_{6k+4}, F_{6k+4}) : k \in \mathbb{N}\}. \tag{3.9}$$

Proof. From Definition 2.5 and parametrization (1.3), we have

$$\begin{aligned}(u', v') \in P'(*, 2, 3|\pm 2) &\Leftrightarrow u'^2 + v'^2 - \tfrac{3}{2}(u'^2 - v'^2) = \pm 2 \\ &\Leftrightarrow u'^2 - 5v'^2 = \mp 4 \\ &\Leftrightarrow u' = L_j,\ v' = F_j\end{aligned} \tag{3.10}$$

where the last line follows from Theorem 3.1. We should also note that j is even if and only if t is negative. We must be careful however in choosing j. In order for the parametrization (1.3) to be valid, $u'_j = L_j$ and $v'_j = F_j$ must both be odd, which is true if and only if j is not divisible by 3. We also note that $j = 1$ does not lead to a valid parametrization since $u'_1 = L_1 = 1$ and $v'_1 = F_1 = 1$ are equal and we require that $u'_j > v'_j$. Nevertheless, every other choice of positive integer j is valid, and it is easy to see that odd j ($j = 6k \pm 1$ for some positive integer k) correspond to elements (u'_j, v'_j) of $P'(*, 2, 3|2)$ and even j ($j = 6k + 2$ or $6k + 4$ for some nonnegative integer k) correspond to elements of $P'(*, 2, 3|-2)$. The result follows. □

It is instructive to look at a table of $u'_j = L_j$, $v'_j = F_j$, $t_j = \frac{1}{2}(u'^2_j - 5v'^2_j) = 2c_j - 3b_j = 2(-1)^{j+1}$ as well as a_j, b_j, and c_j for some small values of j.

j	u'_j	v'_j	a_j	b_j	c_j	t_j
0	2	0	–	–	–	–
1	1	1	–	–	–	–
2	3	1	3	4	5	−2
3	4	2	–	–	–	–
4	7	3	21	20	29	−2
5	11	5	55	48	73	2
6	18	8	–	–	–	–
7	29	13	377	336	505	2
8	47	21	987	884	1325	−2
9	76	34	–	–	–	–

Table 2. The first few elements of the PPTF's $\Pi(*, 2, 3|\pm 2)$ and their parametrizations.

Now to complete the proof of the theorem, we merely note the following easily-derived identities:

$$a_j = u'_j v'_j = L_j F_j = F_{2j}; \tag{3.11}$$

$$b_j = \tfrac{1}{2}(u'^2_j - v'^2_j) = \tfrac{1}{2}(L_j^2 - F_j^2) = \tfrac{2}{5}[L_{2j} + 3(-1)^j]; \tag{3.12}$$

$$c_j = \tfrac{1}{2}(u'^2_j + v'^2_j) = \tfrac{1}{2}(L_j^2 + F_j^2) = \tfrac{1}{5}[3L_{2j} + 4(-1)^j]. \tag{3.13}$$

Equations (3.6) and (3.7) now follow easily from equations (3.8) and (3.9), respectively as well as equations (3.11) - (3.13). □

For completeness, we write down the first few solutions to $\Pi(*, 2, 3|\pm 2)$.

$$\Pi(*, 2, 3|-2) = \{(3, 4, 5), (21, 20, 29), (987, 884, 1325), (6765, 6052, 9077), \ldots\}; \tag{3.14}$$

$$\Pi(*, 2, 3|2) = \{(55, 48, 73), (377, 336, 505), (17711, 15840, 23761), (121393, 108576, 162865) \ldots\}. \tag{3.15}$$

3.2. **Type-1b; $m = 2, n = 3$.** Here we investigate PPTF's of the form $\Pi(2, *, 3|t)$ for some small t. Like the previous case, these PPTF's correspond to right triangles which are asymptotically similar to one with legs of length 2 and $\sqrt{5}$ and hypotenuse of length 3, so we expect the same type of connection with Fibonacci and Lucas numbers. For these PPTF's we have $2c - 3a = t$, hence,

$$2(u^2 + v^2) - 3(u^2 - v^2) = t,$$

or

$$u^2 - 5v^2 = -t.$$

Here the simplest nontrivial case to consider is $t = \pm 1$. The following theorem classifies all PPTF's of the form $\Pi(2, *, 3|\pm 1)$.

Theorem 3.5. *The PPTF's $\Pi(2, *, 3|1)$ and $\Pi(2, *, 3|-1)$ are given by the following formulas:*

$$\Pi(2, *, 3|1) = \{(\tfrac{1}{5}(L_{12k+6} - 3), \tfrac{1}{2}F_{12k+6}, \tfrac{1}{10}(3L_{12k+6} - 4)), k \in \mathbb{N}\}; \tag{3.16}$$

$$\Pi(2, *, 3|-1) = \{(\tfrac{1}{5}(L_{12k} + 3), \tfrac{1}{2}F_{12k}, \tfrac{1}{10}(3L_{12k} + 4)), k \in \mathbb{Z}^+\}. \tag{3.17}$$

Proof. We start by proving the following lemma, from which Theorem 3.5 will easily follow.

Lemma 3.6. *We have*

$$P(2, *, 3|1) = \{(L_{6k+3}, F_{6k+3}) : k \in \mathbb{N}\} \tag{3.18}$$

and

$$P(2, *, 3|-1) = \{(L_{6k}, F_{6k}) : k \in \mathbb{Z}^+\}. \tag{3.19}$$

Proof. From Definition 2.4 and parametrization (1.2), we have

$$\begin{aligned}(u, v) \in P(2, *, 3|\pm 1) &\Leftrightarrow 2(u^2 + v^2) - 3(u^2 - v^2) = \pm 1 \\ &\Leftrightarrow u^2 - 5v^2 = \mp 1 \\ &\Leftrightarrow u = \tfrac{1}{2}L_{3j},\ v = \tfrac{1}{2}F_{3j},\end{aligned} \tag{3.20}$$

where the last line follows from Theorem 3.2. We also have

$$t = 5v^2 - u^2 = (-1)^{j+1} \tag{3.21}$$

hence (3.18) and (3.19) follow. □

Below we tabulate the first few values of $u_j = \frac{1}{2}L_{3j}$, $v_j = \frac{1}{2}F_{3j}$, $a_j = u_j^2 - v_j^2$, $b_j = 2u_jv_j$, $c_j = u_j^2 + v_j^2$, and $t_j = 5v_j - u_j = 2a_j - 3c_j = (-1)^{j+1}$.

j	u_j	v_j	a_j	b_j	c_j	t_j
0	1	0	–	–	–	–
1	2	1	3	4	5	1
2	9	4	65	72	97	−1
3	38	17	1155	1292	1733	1
4	161	72	20737	23184	31105	−1

Table 3. The first few elements of the PPTF's $\Pi(2, *, 3|\pm 1)$ and their parametrizations.

Now to complete the proof of the theorem, we merely note the following identities, which follow from equations (3.11)-(3.13):

$$a_k = u_k^2 - v_k^2 = \tfrac{1}{4}(L_{3k}^2 - F_{3k}^2) = \tfrac{1}{5}[L_{6k} + 3(-1)^k]; \tag{3.22}$$

$$b_k = 2u_k v_k = \tfrac{1}{2}L_{3k}F_{3k} = \tfrac{1}{2}F_{6k}; \tag{3.23}$$

$$c_k = u_k^2 + v_k^2 = \tfrac{1}{4}(L_{3k}^2 + F_{3k}^2) = \tfrac{1}{10}[3L_{6k} + 4(-1)^k]. \tag{3.24}$$

□

As before, we present the first few solutions to $\Pi(2,*,3|\pm 1)$.

$$\Pi(2,*,3|1) = \{(3,4,5),(1155,1292,1733),(372099,416020,558149),\ldots\}; \tag{3.25}$$

$$\Pi(2,*,3|-1) = \{(65,72,97),(20737,23184,31105),(6677057,7465176,10015585),\ldots\}. \tag{3.26}$$

3.3. **Type-2; $m = 1, n = 2$.** In this subsection we consider type-2 PPTF's with $m = 1$ and $n = 2$. Such PPTF's are asymptotically similar to a right triangle with legs of length 1 and 2 and hypotenuse of length $\sqrt{5}$, hence we might expect such a connection with Fibonacci and Lucas numbers once again.

Consider the PPTF $S = \Pi(1,2,*|t)$, where t is a small integer yet to be determined. Every PPT (a,b,c) in S satisfies the equation $b - 2a = t$, which by virtue of (1.2) implies that the parameters u and v satisfy the equation

$$2uv - 2u^2 + 2v^2 = t. \tag{3.27}$$

Since the left side is even, the right side must also be, implying that t is even. The simplest possibilities are $t = \pm 2$.

Theorem 3.7. *We have*

$$\begin{aligned}\Pi(1,2,*|-2) = \{&(\tfrac{1}{5}(L_{12k-7}+4), \tfrac{2}{5}(L_{12k-7}-1), F_{12k-7}) : k \in \mathbb{Z}^+\} \\ \cup\, \{&(\tfrac{1}{5}(L_{12k+1}+4), \tfrac{2}{5}(L_{12k+1}-1), F_{12k+1}) : k \in \mathbb{Z}^+\}\end{aligned} \tag{3.28}$$

and

$$\begin{aligned} lcl\Pi(1,2,*|2) = \{&(\tfrac{1}{5}(L_{12k-5}-4), \tfrac{2}{5}(L_{12k-5}+1), F_{12k-5}) : k \in \mathbb{Z}^+\} \\ \cup\, \{&(\tfrac{1}{5}(L_{12k-1}-4), \tfrac{2}{5}(L_{12k-1}+1), F_{12k-1}) : k \in \mathbb{Z}^+\}.\end{aligned} \tag{3.29}$$

Proof. We start by proving the following lemma, from which Theorem 3.7 will easily follow.

Lemma 3.8. *We have*

$$P(1,2,*|-2) = \{(F_{6k-3}, F_{6k-4}) : k \in \mathbb{Z}^+\} \cup \{(F_{6k+1}, F_{6k}) : k \in \mathbb{Z}^+\} \tag{3.30}$$

and

$$P(1,2,*|2) = \{(F_{6k-2}, F_{6k-3}) : k \in \mathbb{Z}^+\} \cup \{(F_{6k}, F_{6k-1}) : k \in \mathbb{Z}^+\}. \tag{3.31}$$

Proof. After some simplification, (3.27) becomes

$$w^2 - 5v^2 = 2t = \pm 4, \tag{3.32}$$

where $w = 2u - v$. By Theorem 3.1, we have $w = L_j$ and $v = F_j$ for some nonnegative integer j. Thus we have $u = \frac{1}{2}(v + w) = \frac{1}{2}(L_j + F_j) = F_{j+1}$ and $v = F_j$.

The following table lists the first few values of $u_j = F_{j+1}$ and $v_j = F_j$ as well as a_j, b_j, c_j, and $t_j = b_j - 2a_j = 2(-1)^{j+1}$ where applicable.

j	u_j	v_j	a_j	b_j	c_j	t_j
1	1	1	–	–	–	–
2	2	1	3	4	5	−2
3	3	2	5	12	13	2
4	5	3	–	–	–	–
5	8	5	39	80	89	2
6	13	8	105	208	233	−2
7	21	13	–	–	–	–

Table 3. The first few elements of the PPTF's $\Pi(1,2,*|\pm 2)$ and their parametrizations.

The rows in which a_j, b_j, c_j, and t_j are not listed are precisely those rows in which u_j and v_j have the same parity (both odd), in which case the parametrization in (1.2) is not applicable. It is easy to see that these rows correspond to values of j congruent to 1 modulo 3. (Alternatively, one can see this by noting that F_j is even if and only if j is divisible by 3, so that $u_j = F_{j+1}$ and $v_j = F_j$ are both odd if and only if j is congruent to 1 modulo 3.)

To complete the proof of the lemma, we note that $t_j = 2$ if and only if j is congruent to 3 or 5 modulo 6, in which case the Fibonacci index of u_j is congruent to -2 (resp. 0) modulo 6 and that of v_j is congruent to -3 (resp. -1) mod 6. □

Now to complete the proof of the theorem, we need to verify that a_j, b_j, and c_j are given by the formulas implied by (3.28) and (3.29), as the case may be. This is equivalent to showing the following:

$$F_{6k+s}^2 - F_{6k+s-1}^2 = \tfrac{1}{5}(L_{12+2s-1} + 4); \tag{3.33}$$

$$F_{6k+s}F_{6k+s-1} = \tfrac{1}{5}(L_{12k+2s-1} - 1); \tag{3.34}$$

$$F_{6k+s}^2 + F_{6k+s-1}^2 = F_{12k+2s-1} \tag{3.35}$$

for all positive integers k and for $s = -3, -2, 0$ and 1. Each of these equations are in fact true for all integers s. Equation (3.34) follows from the identity

$$F_k F_{k+1} = \tfrac{1}{5}(L_{2k+1} - 1), \tag{3.36}$$

which is easily verifiable either by induction or by use of the Binet formulas. Similarly, equation (3.35) follows from the well-known identity [4]

$$F_{k+1}^2 + F_k^2 = F_{2k+1}, \tag{3.37}$$

while (3.33) follows from the easily verifiable identity

$$F_{k+1}^2 - F_k^2 = \tfrac{1}{5}(L_{2k+1} + 4). \tag{3.38}$$

□

For completeness, we write down the first few solutions to $\Pi(1,2,*|\pm 2)$.

$$\Pi(1,2,*|-2) = \{(3,4,5),(105,208,233),(715,1428,1597),(33553,67104,75025),\ldots\}. \tag{3.39}$$

$$\Pi(1,2,*|2) = \{(5,12,13),(39,80,89),(1869,3740,4181),(12815,25632,28657),\ldots\}. \tag{3.40}$$

3.4. **Type-2; $m = 2, n = 1$.** Now let us look at PPTF's of the form $\Pi(2,1,*|t)$ for some small t. As in the previous subsection, such PPTF's are also asymptotically similar to a right triangle with sides of length 2, 1, and $\sqrt{5}$, so we might expect the coefficients of the PPT's in such a PPTF to once again be simply expressible in terms of Fibonacci and Lucas numbers. Each PPT in this class of PPTF's must satisfy $2b-a=t$. Using the alternate parametrization given by (1.3), this becomes

$$u'^2 - v'^2 - u'v' = t. \tag{3.41}$$

Multiplying (3.41) by 4 and completing the square yields

$$(2u' - v')^2 - 5v'^2 = 4t. \tag{3.42}$$

Now if we let $w' = 2u' - v'$, we obtain

$$w'^2 - 5v'^2 = 4t. \tag{3.43}$$

Note that apart from the names of the variables and the value of the right side of the equation, (3.43) is identical to (3.32), which we know how to solve. The main and only difference here is in the value of t, the simplest values now being $t = \pm 1$, which yields

$$w'^2 - 5v'^2 = \pm 4. \tag{3.44}$$

Now this is identical to (3.32), which we solved in the previous subsection. Also analogously to the previous case, we have $u' = \frac{1}{2}(v' + w')$. Thus we have the same solutions for u', v' and w' as we had for u, v, and w, respectively in the previous subsection. Note however that this time, since we are using the alternate parametrization (1.3), u' and v' must both be odd in order to yield a solution. Without much extra work, we can now prove the following theorem.

Theorem 3.9. *We have*

$$\Pi(2,1,*|1) = \{(\tfrac{1}{5}(L_{12k-3} - 1), \tfrac{1}{10}(L_{12k-3} + 4), \tfrac{1}{2}F_{12k-3}) : k \in \mathbb{Z}^+\} \tag{3.45}$$

and

$$\Pi(2,1,*|-1) = \{(\tfrac{1}{5}(L_{12k+3} + 1), \tfrac{1}{10}(L_{12k+3} - 4), \tfrac{1}{2}F_{12k+3}) : k \in \mathbb{Z}^+\}. \tag{3.46}$$

Proof. Once again we begin by first proving a lemma regarding related PPTFP's.

Lemma 3.10. *We have*

$$P'(2,1,*|1) = \{(F_{6k-1}, F_{6k-2}) : k \in \mathbb{Z}^+\} \tag{3.47}$$

and

$$P'(2,1,*|-1) = \{(F_{6k+2}, F_{6k+1}) : k \in \mathbb{Z}^+\}. \tag{3.48}$$

Proof. As noted above, since u' and v' satisfy the same equations as u and v in the previous subsection, they yield the same solutions. However, this time we only obtain PPTF's if u' and v' are both odd. The following table illustrates this:

j	u'_j	v'_j	a_j	b_j	c_j	t_j
1	1	1	–	–	–	–
2	2	1	–	–	–	–
3	3	2	–	–	–	–
4	5	3	15	8	17	1
5	8	5	–	–	–	–
6	13	8	–	–	–	–
7	21	13	273	136	305	−1

Table 4. The first few elements of the PPTF's $\Pi'(2,1,*|\pm 1)$ and their parametrizations.

Note that $u'_j = F_{j+1}$ and $v'_j = F_j$ are both odd if and only if j is congruent to 1 modulo 3. We also note that $t_j = (-1)^j$, which is easy to see by checking the base case and noting that t_j alternates sign as j increases by 3. Thus we see that $P'(2,1,*|1)$ corresponds to the rows of the above table with j congruent to 4 modulo 6, while $P'(2,1,*|1)$ corresponds to the rows with j congruent to 1 modulo 6. The result follows. □

Now to complete the proof of the theorem, we merely apply equations (3.33)-(3.35) again to the equations for a, b, and c, which we obtain by applying equations (1.3). Specifically, for $s = 0$ or 1 we have

$$\begin{array}{lllllll} a & = & u'_{6k-3s}v'_{6k-3s} & = & F_{6k-3s+2}F_{6k-3s+1} & = & \frac{1}{5}(L_{12k-6s+3} + (-1)^s); \\ b & = & \frac{1}{2}({u'_{6k-3s}}^2 - {v'_{6k-3s}}^2) & = & \frac{1}{2}(F^2_{6k-3s+2} - F^2_{6k-3s+1}) & = & \frac{1}{10}(L_{12k-6s+3} + 4(-1)^{s+1}); \\ c & = & \frac{1}{2}({u'_{6k-3s}}^2 + {v'_{6k-3s}}^2) & = & \frac{1}{2}(F^2_{6k-3s+2} + F^2_{6k-3s+1}) & = & \frac{1}{2}F_{12k-6s+3}. \end{array} \tag{3.49}$$

Clearly the case $s = 0$ corresponds to $t = -1$ and $s = 1$ corresponds to $t = 1$. Equations (3.45) and (3.46) thus follow from the above equations. □

Here we list the first few solutions to $\Pi(2,1,*|\pm 1)$.

$$\Pi(2,1,*|1) = \{(15,8,17), (4895,2448,5473), (1576239,788120,1762289), \ldots\}. \tag{3.50}$$

$$\Pi(2,1,*|-1) = \{(273,136,305), (87841,43920,98209), (28284465,14142232,31622993), \ldots\}. \tag{3.51}$$

4. Further Work

We seem to be touching on an exciting topic, which clearly can be expanded. For instance, it would be useful to know if we can write down solutions to all PPTF's, or at least to those with small values of t. The connection with Fibonacci and Lucas numbers is quite intriguing. Perhaps there are such connections as well for other values of m and n and/or for other types. Similarly, the connection with Pell numbers and companion Pell numbers for type-2 PPTF's with $m = n = 1$ suggests that there may be similar connections for other values of m and n and/or for other types. In fact, we may be able to generalize to sequences such as these for arbitrary PPTF types and arbitrary m and n. We may even be able to go further. Why restrict ourselves to right triangles? It may be worthwhile to investigate arbitrary triangles in which one of the angles is some fixed value with a rational cosine.

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MSC2010: 11B39, 11R11

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