PROOF OF KIMBERLING'S "EVEN SECOND COLUMN" CONJECTURE

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Abstract. The quasi-Zeckendorf representations of a positive integer are introduced, and are used to prove a conjecture of Kimberling about a certain Stolarsky interspersion (the "even second column" array). In addition, alternative proofs of some known results about Stolarsky interspersions are given.

1. Introduction

The "even second column" (ESC) array was introduced by Clark Kimberling in [1], as an example of a *Stolarsky interspersion*. The following characterization of Stolarsky interspersions is taken from [1] (with changed notation in particular, indexing here starts at 0 not 1). Let (δ_i) $(i \geq 0)$ be any sequence with $\delta_0 = 1$ and $\delta_i = 0$ or 1 $(i \geq 1)$. The Stolarsky interspersion corresponding to (δ_i) is an array $a(i,j)$ $(i, j \geq 0)$ defined by induction on the rows, as follows.

For integer $i \geq 0$, let $a(i,0)$ be the least positive integer that has not already occurred in rows $0, \ldots, i-1$ (so that $a(0,0) = 1$). Let $a(i,1) = \lfloor \Phi a(i,0) \rfloor + \delta_i$, where $\Phi = (\sqrt{5}+1)/2 = 1.618\ldots$. Continue the row by the Fibonacci-type recurrence $a(i,j) = a(i,j-1) + a(i,j-2)$ $(j \geq 2)$. Thus row 0 consists of the Fibonacci sequence $1, 2, 3, 5, 8, \ldots$; and row 1 begins $4, 6, 10, \ldots$ or $4, 7, 11, \ldots$ according as $\delta_i = 0$ or 1.

The ESC array is obtained by taking $\delta_i = 1$ if i is even, 0 if i is odd. See Table 1 below for the top left corner of the ESC array. The main object of this paper is to prove Kimberling's conjecture [1] that every number in the second column of the ESC array is even. It is also shown how the techniques developed here lead to alternative proofs of some known results on Stolarsky interspersions.

Given a row containing an integer n, say $n = a(i,j)$, for integer $p \geq -j$ let $\mathrm{S}_p(n)$ denote $a(i, j+p)$, the pth successor of n in its row. If $j > 0$ then $\mathrm{S}_{-1}(n)$ is the predecessor of n.

2. Zeckendorf Representation of a Positive Integer

Theorem 2.1. *Every positive integer n can be represented uniquely as a sum of Fibonacci numbers*

$$n = F_{c_0} + F_{c_1} + \cdots + F_{c_q} \tag{2.1}$$

such that

(i) $c_i - c_{i-1} \geq 2$ for $i = 1, \ldots, q$,
(ii) $c_0 \geq 2$.

This is *Zeckendorf's Theorem* [5] (see also [3]) and such a representation is called a *Zeckendorf representation*, here abbreviated to ZRep.

Lemma 2.2. *For any integer r,*

(i) $\Phi F_r = F_{r+1} - (-\Phi)^{-r}$,
(ii) $\Phi^{-1} F_r = F_{r-1} - (-\Phi)^{-r}$,
(iii) $\Phi^{-2} F_r = F_{r-2} + (-\Phi)^{-r}$.

Proof. These follow easily from Binet's formula $F_r = (\Phi^r - (-\Phi)^{-r})/\sqrt{5}$. $\square$

Lemma 2.3. *Let $c_0, c_1, \ldots, c_q$ be integers with $c_i - c_{i-1} \geq 2$ $(i = 1, \ldots, q)$. If p is an integer such that $0 \leq p \leq q$ then*

$$(-\Phi)^{-c_p} + (-\Phi)^{-c_{p+1}} + \cdots + (-\Phi)^{-c_q} = (-\Phi)^{-c_p}\theta, \tag{2.2}$$

where $\Phi^{-1} < \theta < \Phi$.

Proof. Regard (2.2) as the definition of θ, and use backwards induction on p. If $p = q$ then $\theta = 1$ and the result holds. Suppose $0 \leq p < q$ and the result holds with $p + 1$ in place of p. For convenience, write $c = c_p,\ d = c_{p+1}$. By the inductive hypothesis,

$$\theta = 1 + (-\Phi)^{c-d}\theta', \quad \text{where } \Phi^{-1} < \theta' < \Phi.$$

If $c \equiv d \pmod 2$ then $\Phi^{-1} < 1 < \theta \leq 1 + \Phi^{-2}\theta' < 1 + \Phi^{-1} = \Phi$.
If $c \not\equiv d \pmod 2$ then $d - c \geq 3$, so $\Phi > 1 > \theta \geq 1 - \Phi^{-3}\theta' > 1 - \Phi^{-2} = \Phi^{-1}$.
In either case, the inductive step is proved. $\square$

Proposition 2.4. *Let n be a positive integer and let (2.1) be the ZRep of n. Then*

$$\begin{aligned} \Phi^{-c_0-1} < \{\Phi^{-2}n\} &< \Phi^{-c_0+1} && \text{if } c_0 \text{ is even} \\ 1 - \Phi^{-c_0+1} < \{\Phi^{-2}n\} &< 1 - \Phi^{-c_0-1} && \text{if } c_0 \text{ is odd,} \end{aligned}$$

where $\{\ \}$ denotes the fractional part.

Proof. By Lemma 2.2(iii), $\Phi^{-2}n = m + \xi$ where

$$m = F_{c_0-2} + F_{c_1-2} + \cdots + F_{c_q-2}, \quad \xi = (-\Phi)^{-c_0} + (-\Phi)^{-c_1} + \cdots + (-\Phi)^{-c_q}.$$

Lemma 2.3 with $p = 0$ gives $\xi = (-\Phi)^{-c_0}\theta$ where $\Phi^{-1} < \theta < \Phi$. By the definition of a ZRep, $c_0 \geq 2$, so $|\xi| \leq \Phi^{-1}$. We have

$$\Phi^{-2}n = \begin{cases} m + |\xi| & \text{if } c_0 \text{ is even} \\ (m-1) + (1 - |\xi|) & \text{if } c_0 \text{ is odd,} \end{cases} \tag{2.3}$$

and the result follows. $\square$

Since in Proposition 2.4 the intervals for different values of c_0 are disjoint, Proposition 2.4 can be applied in reverse to determine c_0 from $\{\Phi^{-2}n\}$. This is done at the start of Section 4.

3. Quasi-Zeckendorf Representations of a Positive Integer

Define a *quasi-Zeckendorf representation* (QZRep) to be a representation (2.1) where

(i) $c_i - c_{i-1} \geq 2$ for $i = 1, \ldots, q$ (as in a ZRep),
(ii) $c_0 \geq 0$ (i.e. F_0 and F_1 are now allowed),
(iii) if $c_0 = 0$, then c_1 (necessarily present) is odd.

Clearly every ZRep is a QZRep. For example $F_1 + F_4$ and $F_2 + F_4$ are to be considered distinct QZReps of 4, although $F_1 = F_2$. A QZRep will be called odd or even if its least index c_0 is odd or even.

In the following, $\langle c_0, c_1, \ldots, c_q \rangle$ denotes a QZRep

$$F_{c_0} + F_{c_1} + \cdots + F_{c_q} \quad (c_0 < c_1 < \cdots < c_q) \tag{3.1}$$

and $\mathrm{val}\langle c_0, c_1, \ldots, c_q \rangle$ denotes the integer that it represents.

Proposition 3.1. *Every positive integer n has a unique odd QZRep and a unique even QZRep.*

1	2	3	5	8	...	$\langle 2\rangle$	$\langle 3\rangle$	$\langle 4\rangle$	$\langle 5\rangle$	$\langle 6\rangle$	...
4	6	10	16	26	...	$\langle 1,4\rangle$	$\langle 2,5\rangle$	$\langle 3,6\rangle$	$\langle 4,7\rangle$	$\langle 5,8\rangle$	...
7	12	19	31	50	...	$\langle 0,3,5\rangle$	$\langle 1,4,6\rangle$	$\langle 2,5,7\rangle$	$\langle 3,6,8\rangle$	$\langle 4,7,9\rangle$	...
9	14	23	37	60	...	$\langle 1,6\rangle$	$\langle 2,7\rangle$	$\langle 3,8\rangle$	$\langle 4,9\rangle$	$\langle 5,10\rangle$	...
11	18	29	47	76	...	$\langle 4,6\rangle$	$\langle 5,7\rangle$	$\langle 6,8\rangle$	$\langle 7,9\rangle$	$\langle 8,10\rangle$	...
...						...					

Table 1: Top left of the ESC array with (right) QZRep notation.

Proof. Existence. If the ZRep of n is odd, it is an odd QZRep, and an even QZRep is obtained by prefixing F_0. If the ZRep of n is even, it is an even QZRep, and if the smallest index is $2t$ then an odd QZRep is obtained from the (easily proved) relation

$$F_1 + F_3 + \cdots + F_{2t-1} = F_{2t} \quad (t = 1, 2, 3, \ldots). \tag{3.2}$$

Uniqueness. Suppose n has two odd QZReps. If neither includes F_1 they are ZReps and thus identical by Zeckendorf's Theorem 2.1. If both include F_1 then either $n = 1$ (a trivial case), or we can delete F_1 to obtain two ZReps of $n - 1$ and again apply Theorem 2.1. If the first includes F_1 and the second does not, suppose the first includes $F_1, \ldots, F_{2t-1}$ but not F_{2t+1}, and replace these by F_{2t}; this gives both an even and an odd ZRep of n; a contradiction. Suppose n has two even QZReps. If neither includes F_0 they are identical by Theorem 2.1. If both include F_0, we can delete F_0 and again apply Theorem 2.1. If one includes F_0 and the other does not, delete F_0 to obtain both an even and an odd ZRep of n; a contradiction. □

Proposition 3.2. *Let n be a positive integer. If every Fibonacci index in the odd (resp. even) QZRep of n is increased by 1, the result is a QZRep of $\lfloor \Phi n \rfloor$ (resp. $\lfloor \Phi n \rfloor + 1$).*

Proof. Clearly the result is a QZRep. Let $n = \mathrm{val}\langle c_0, c_1, \ldots, c_q\rangle$ be a QZRep of n, and define

$$n^+ = \mathrm{val}\langle c_0 + 1, c_1 + 1, \ldots, c_q + 1\rangle. \tag{3.3}$$

By Lemma 2.2(i), $n^+ - \Phi n = (-\Phi)^{-c_0} + \cdots + (-\Phi)^{-c_q}$, hence by Lemma 2.3 with $p = 0$,

$$n^+ - \Phi n = (-\Phi)^{-c_0}\theta, \quad \text{where } \Phi^{-1} < \theta < \Phi.$$

Case 1. If c_0 is odd then $0 < \Phi n - n^+ < \Phi^{1-c_0} \le 1$, so $n^+ = \lfloor \Phi n \rfloor$.

Case 2. If c_0 is even and $c_0 > 0$ then $0 < n^+ - \Phi n < \Phi^{1-c_0} < 1$, so $n^+ = \lfloor \Phi n \rfloor + 1$.

Case 3. If $c_0 = 0$ then F_{c_1} is present with c_1 odd, so by Case 1 we have $n^+ = F_1 + \lfloor \Phi n \rfloor = \lfloor \Phi n \rfloor + 1$. □

Proposition 3.2 can be applied to generate the Stolarsky interspersion corresponding to a given zero–one sequence (δ_i). Having found the first term of row i, write down its odd QZRep if $\delta_i = 0$ or its even QZRep if $\delta_i = 1$. By Proposition 3.2 the second term of the row is obtained by incrementing the indices by 1, and then the Fibonacci-type rule implies that the rest of the row is generated by further increments. This method of "shifting subscripts" was used by Kimberling [2] to generate a particular Stolarsky interspersion (the Wythoff array). Our Proposition 3.2 is similar to Lemma 2 in [2, p. 4].

Table 1 shows the top left corner of the ESC array, as generated by this method.

Given a QZRep $\langle c_0, \ldots, c_q\rangle$, define $\mathrm{T}_j\langle c_0, \ldots, c_q\rangle$ $(j = 0, 1, 2)$ to be the number of indices $c_0, \ldots, c_q$ that are *not* congruent to j (mod 3).

Proposition 3.3. *For any QZRep $\langle c_0, \ldots, c_q\rangle$, we have*

(i) $\mathrm{T}_0\langle c_0, \dots, c_q\rangle \equiv \mathrm{val}\langle c_0, \dots, c_q\rangle \pmod 2$.
(ii) $\mathrm{T}_2\langle c_0, \dots, c_q\rangle \equiv \mathrm{val}\langle c_0 + 1, \dots, c_q + 1\rangle \pmod 2$.
(iii) $\mathrm{T}_2\langle c_0, \dots, c_q\rangle + \mathrm{T}_2\langle c_0 + 1, \dots, c_q + 1\rangle + \mathrm{T}_2\langle c_0 + 2, \dots, c_q + 2\rangle \equiv 0 \pmod 2$.
(iv) $\mathrm{T}_0\langle c_0, \dots, c_q\rangle + \mathrm{T}_1\langle c_0, \dots, c_q\rangle + \mathrm{T}_2\langle c_0, \dots, c_q\rangle \equiv 0 \pmod 2$.

Proof. (i) holds because F_r is even if and only if r is divisible by 3. (ii) follows easily from (i). (iii) and (iv) hold because the LHS counts each c_r twice. □

4. Blocks of Integers: Types A, B, C

Partition the positive integers into blocks separated by multiples of $\Phi^2 = 2.618\dots$, and label the blocks 0, 1, 2, ... Thus, n belongs to block i if and only if $\lfloor \Phi^{-2} n \rfloor = i$.

Proposition 4.1. *Let Q be a QZRep. Then $\mathrm{T}_2 Q$ is even if and only if Q is either the even QZRep of an integer in an even block or the odd QZRep of an integer in an odd block.*

Proof. Let $Q = \langle c_0, \dots, c_q\rangle$ and $n = \mathrm{val}Q$. Let n be in block i; then by the way the blocks are defined, $i = \lfloor \Phi^{-2} n \rfloor$. Since $\Phi^{-2} = 1 - \Phi^{-1}$ and $\Phi = 1 + \Phi^{-1}$, we have

$$i = \lfloor \Phi^{-2} n \rfloor = n - \lfloor \Phi^{-1} n \rfloor - 1 \not\equiv n + \lfloor \Phi^{-1} n \rfloor = \lfloor \Phi n \rfloor \pmod 2. \tag{4.1}$$

Let $n^+ = \mathrm{val}\langle c_0 + 1, \dots, c_q + 1\rangle$. By Proposition 3.3(ii), $\mathrm{T}_2\langle c_0, \dots, c_q\rangle \equiv n^+ \pmod 2$. There are four cases if i and c_0 are independently even or odd. If for example both are even, then by (4.1) $\lfloor \Phi n \rfloor$ is odd, and by Proposition 3.2 $n^+ = \lfloor \Phi n \rfloor + 1$, so n^+ is even. The other three cases are similar. □

		n	Odd QZRep of n	Even QZRep of n
Block 0	*	1	$\langle\mathbf{1}\rangle$	$\langle 2\rangle$
Type A		2	$\langle\mathbf{3}\rangle$	$\langle\mathbf{0},\mathbf{3}\rangle$
Block 1		3	$\langle\mathbf{1},\mathbf{3}\rangle$	$\langle\mathbf{4}\rangle$
Type B	*	4	$\langle\mathbf{1},\mathbf{4}\rangle$	$\langle 2,\mathbf{4}\rangle$
		5	$\langle 5\rangle$	$\langle\mathbf{0},5\rangle$
Block 2		6	$\langle\mathbf{1},5\rangle$	$\langle 2,5\rangle$
Type A	*	7	$\langle\mathbf{3},5\rangle$	$\langle\mathbf{0},\mathbf{3},5\rangle$
Block 3		8	$\langle\mathbf{1},\mathbf{3},5\rangle$	$\langle\mathbf{6}\rangle$
Type C	*	9	$\langle\mathbf{1},\mathbf{6}\rangle$	$\langle 2,\mathbf{6}\rangle$
		10	$\langle\mathbf{3},\mathbf{6}\rangle$	$\langle\mathbf{0},\mathbf{3},\mathbf{6}\rangle$
Block 4	*	11	$\langle\mathbf{1},\mathbf{3},\mathbf{6}\rangle$	$\langle\mathbf{4},\mathbf{6}\rangle$
Type B		12	$\langle\mathbf{1},\mathbf{4},\mathbf{6}\rangle$	$\langle 2,\mathbf{4},\mathbf{6}\rangle$
		13	$\langle\mathbf{7}\rangle$	$\langle\mathbf{0},\mathbf{7}\rangle$
Block 5		14	$\langle\mathbf{1},\mathbf{7}\rangle$	$\langle 2,\mathbf{7}\rangle$
Type A	*	15	$\langle\mathbf{3},\mathbf{7}\rangle$	$\langle\mathbf{0},\mathbf{3},\mathbf{7}\rangle$
Block 6		16	$\langle\mathbf{1},\mathbf{3},\mathbf{7}\rangle$	$\langle\mathbf{4},\mathbf{7}\rangle$
Type B	*	17	$\langle\mathbf{1},\mathbf{4},\mathbf{7}\rangle$	$\langle 2,\mathbf{4},\mathbf{7}\rangle$
		18	$\langle 5,\mathbf{7}\rangle$	$\langle\mathbf{0},5,\mathbf{7}\rangle$
Block 7		19	$\langle\mathbf{1},5,\mathbf{7}\rangle$	$\langle 2,5,\mathbf{7}\rangle$
Type A	*	20	$\langle\mathbf{3},5,\mathbf{7}\rangle$	$\langle\mathbf{0},\mathbf{3},5,\mathbf{7}\rangle$

Table 2: QZReps of integers and partitioning into blocks.

Table 2 shows the first 8 blocks, with the QZReps of each n. To illustrate Proposition 4.1, indices congruent to 0 or 1 $\pmod 3$ are shown in bold. Integers that start a row in the ESC array are marked with an asterisk. We see empirically that there is exactly one such integer in each block; this is true for all Stolarsky interspersions, as will be proved in Proposition 4.4.

The block types A, B, C will now be defined. Let b be the first integer in a block, and let $\xi = \{\Phi^{-2}b\}$. Since b starts a block, $\xi \le \Phi^{-2}$, with equality if and (since Φ^{-2} is irrational) only if $b = 1$. Proposition 2.4 shows that $\xi \ne \Phi^{-r}$ for odd r. The block type is defined in terms of ξ as follows.

Type A. $\Phi^{-3} < \xi \le \Phi^{-2}$. Since $\xi + 2\Phi^{-2} > \Phi^{-3} + 2\Phi^{-2} = 1$, the block has only 2 terms. Block 0, consisting of 1 and 2, is of this type. Now assume $b > 2$. By Proposition 2.4, the ZRep of b begins $\langle 2, \ldots\rangle$. To get the next index, note that

$$\{\Phi^{-2}(b - F_2)\} = 1 + \xi - \Phi^{-2} > 1 + \Phi^{-3} - \Phi^{-2} = 1 - \Phi^{-4} \tag{4.2}$$

so by Proposition 2.4 the ZRep of b begins $\langle 2, d, \ldots\rangle$ with odd $d \ge 5$. The ZRep of $b + 1$ therefore begins $\langle 3, d, \ldots\rangle$.

Type B. $\Phi^{-5} < \xi < \Phi^{-3}$. Since $\xi + 2\Phi^{-2} < 1$, the block has 3 terms. By Proposition 2.4 the ZRep of b begins $\langle 4, f, \ldots\rangle$, where any $f \ge 6$ can occur. Hence the ZRep of $b + 1$ begins $\langle 2, 4, f, \ldots\rangle$ and the ZRep of $b+2$ begins $\langle 5, f, \ldots\rangle$ if $f > 6$, or $\langle d, \ldots\rangle$ with odd $d \ge 7$ if $f = 6$.

Type C. $\xi < \Phi^{-5}$. The block has 3 terms. By Proposition 2.4 the ZRep of b begins $\langle e, \ldots\rangle$ with even $e \ge 6$. Hence the ZRep of $b + 1$ begins $\langle 2, e, \ldots\rangle$, and the ZRep of $b + 2$ begins $\langle 3, e, \ldots\rangle$.

The QZReps of the integers in each type of block are shown in Table 3.

In the rest of this paper, b_i denotes the first term in block i. All Stolarsky interspersions below are assumed to be generated in terms of QZReps by the method of increasing indices, as explained after Proposition 3.2. It is important to consider not only the value of an element $a(i, j)$, but also the QZRep by which $a(i, j)$ is generated. For this purpose it will often be necessary to refer to Table 3.

Lemma 4.2. *In a Stolarsky interspersion generated by QZReps:*

(i) A QZRep $\langle c_0, c_1, \ldots\rangle$ with $c_0 \equiv c_1 \pmod 2$ can occur only in columns $0, 1, \ldots, c_0 - 1$.
(ii) For all $i \ge 0$, $b_i + 1$ can occur only in columns 0, 1, 2.
(iii) In row 0, each QZRep consists of a single index.
(iv) In row 0, apart from the first 3 terms b_0, $b_0 + 1$, b_1, each term is either $b_i + 2$ for some block i of type B, or b_i for some block i of type C.

Proof. (i) Clearly any QZRep $\langle c_0, c_1, \ldots\rangle$ can occur only in columns $0, 1, \ldots, c_0$. If it occurs in column c_0 then the row starts with $\langle 0, c_1 - c_0, \ldots\rangle$, so by (iii) in the definition of a QZRep $c_0 \not\equiv c_1 \pmod 2$. (ii) follows from (i) and inspection of Table 3.

(iii) We always have $a(0, 0) = 1$ and $\delta_0 = 1$, so row 0 begins with $\langle 2\rangle$, the even QZRep of 1, and continues $\langle 3\rangle, \langle 4\rangle, \ldots$. (iv) follows from (iii) and the inspection of Table 3. □

The next lemma (4.3) summarizes results about the successors and predecessors of an element in a Stolarsky interspersion. For block number $i > 1$ let d, e, f be as in Table 3. Depending on the type of block i, define u_i, v_i, w_i as follows (v_i and w_i are undefined when block i is of

Type A Block i has 2 terms, with QZReps (odd $d \geq 5$):

$$b_i = \mathrm{val}\langle 1, d, \ldots\rangle = \mathrm{val}\langle 2, d, \ldots\rangle$$
$$b_i + 1 = \mathrm{val}\langle 3, d, \ldots\rangle = \mathrm{val}\langle 0, 3, d, \ldots\rangle,$$

except that if $i = 0$ then the terms $d, \ldots$ are absent.

Type B Block i has 3 terms with QZReps (any $f \geq 6$):

$$b_i = \mathrm{val}\langle 1, 3, f, \ldots\rangle = \mathrm{val}\langle 4, f, \ldots\rangle$$
$$b_i + 1 = \mathrm{val}\langle 1, 4, f, \ldots\rangle = \mathrm{val}\langle 2, 4, f, \ldots\rangle$$
$$b_i + 2 = \begin{cases} \mathrm{val}\langle 5, f, \ldots\rangle = \mathrm{val}\langle 0, 5, f, \ldots\rangle & \text{if } f > 6 \\ \mathrm{val}\langle d, \ldots\rangle = \mathrm{val}\langle 0, d, \ldots\rangle \text{ (odd } d \geq 7) & \text{if } f = 6, \end{cases}$$

except that if $i = 1$ then the terms $f, \ldots$ are absent.
If $f = 6$ then d is the greatest odd integer such that $4, 6, \ldots, d-1$ occur in the even QZRep of b_i.

Type C Block i has 3 terms with QZReps (even $e \geq 6$):

$$b_i = \mathrm{val}\langle 1, 3, \ldots, e-1, \ldots\rangle = \mathrm{val}\langle e, \ldots\rangle$$
$$b_i + 1 = \mathrm{val}\langle 1, e, \ldots\rangle = \mathrm{val}\langle 2, e, \ldots\rangle$$
$$b_i + 2 = \mathrm{val}\langle 3, e, \ldots\rangle = \mathrm{val}\langle 0, 3, e, \ldots\rangle.$$

Table 3: Odd and even QZReps of integers in block i.

type A):

$$u_i = \begin{cases} \mathrm{val}\langle 1, d-1, \ldots\rangle = \mathrm{val}\langle 2, d-1, \ldots\rangle & \text{(Type A)} \\ \mathrm{val}\langle 3, f-1, \ldots\rangle = \mathrm{val}\langle 0, 3, f-1, \ldots\rangle & \text{(Type B)} \\ \mathrm{val}\langle e-1, \ldots\rangle = \mathrm{val}\langle 0, e-1, \ldots\rangle & \text{(Type C).} \end{cases} \tag{4.3}$$

$$v_i = \begin{cases} \mathrm{val}\langle 1, 3, f-1, \ldots\rangle = \mathrm{val}\langle 4, f-1, \ldots\rangle & \text{(Type B, } f > 6) \\ \mathrm{val}\langle 1, 3, 5, \ldots, d-2, \ldots\rangle = \mathrm{val}\langle d-1, \ldots\rangle & \text{(Type B, } f = 6) \\ \mathrm{val}\langle 1, e-1, \ldots\rangle = \mathrm{val}\langle 2, e-1, \ldots\rangle & \text{(Type C).} \end{cases} \tag{4.4}$$

$$w_i = \begin{cases} \mathrm{val}\langle 1, f-3, \ldots\rangle = \mathrm{val}\langle 2, f-3, \ldots\rangle & \text{(Type B, } f > 6) \\ \mathrm{val}\langle 1, 3, , \ldots, d-4, \ldots\rangle = \mathrm{val}\langle d-3, \ldots\rangle & \text{(Type B, } f = 6) \\ \mathrm{val}\langle e-3, \ldots\rangle = \mathrm{val}\langle 0, e-3, \ldots\rangle & \text{(Type C).} \end{cases} \tag{4.5}$$

Lemma 4.3. *For $i > 1$ let u_i, v_i, w_i be defined as above.*

(i) Successors: If u_i occurs in a Stolarsky interspersion then $\mathrm{S}_1(u_i) = b_i$ or b_i+1 if u_i occurs in its odd or even QZRep. Similarly, when block i is of type B or C, $\mathrm{S}_1(v_i) = b_i + 1$ or $b_i + 2$ and $\mathrm{S}_3(w_i) = b_i$ or $b_i + 2$.

(ii) Predecessors: If b_i occurs in a Stolarsky interspersion, and not in column 0, then $\mathrm{S}_{-1}(b_i) = u_i$. Similarly, $\mathrm{S}_{-1}(b_i + 1) = u_i$ or v_i and $\mathrm{S}_{-1}(b_i + 2) = v_i$.

Proof. The proof follows by routine checking of cases from Table 3. For example, take the two elements of block i when block i is of type A. If b_i is not in column 0 then it occurs in the QZRep $\langle 2, d, \dots\rangle$ not $\langle 1, d, \dots\rangle$ by Lemma 4.2(i), and hence $\mathrm{S}_{-1}(b_i) = \mathrm{val}\langle 1, d-1, \dots\rangle = u_i$. Similarly if b_i+1 is not in column 0 then it occurs as $\langle 3, d, \dots\rangle$ not $\langle 0, 3, d, \dots\rangle$ and $\mathrm{S}_{-1}(b_i+1) = \mathrm{val}\langle 2, d-1, \dots\rangle = u_i$ also. □

The following proposition shows that not only does column 0 of a Stolarsky interspersion contain exactly one representative of each block, but so does the union of columns 1 and 2.

Proposition 4.4. *Suppose a Stolarsky interspersion is generated in terms of QZReps as described above. For integer $i \geq 0$:*

(i) Each integer $1, \dots, b_{i+1}-1$ occurs exactly once in rows $0, \dots, i$.
(ii) $a(i,0)$ belongs to block i.
(iii) Block i contains exactly one integer in column 1 or 2.

Proof. By induction on i. The result is easily checked for $i = 0, 1$. Suppose that $i > 1$ and (i)–(iii) hold with $i-1$ in place of i. The inductive step depends on the type of block i. In the following, d, e, f are as in Table 3.

Type A. Define u_i as in equation (4.3). Then $u < b_i$, so by (i) u_i occurs exactly once in rows $0, \dots, i-1$, and by Proposition 4.3(i) $\mathrm{S}_1(u_i) = b_i$ or b_i+1. Hence, at least one of b_i, b_i+1 occurs in rows $0, \dots, i-1$. Suppose, for a contradiction, that both occur. By (ii) neither starts a row, and by Proposition 4.3(ii) the predecessor of each is u_i. Thus, u occurs twice in rows $0, \dots, i-1$, contrary to (i). Hence, exactly one of b_i, b_i+1 occurs in rows $0, \dots, i-1$, and the other is chosen as $a(i,0)$.

For (iii), there are two cases.

Case $a(i,0) = b_i$. Then b_i+1 occurs in rows $0, \dots, i-1$, and not in column 0 by inductive hypothesis (ii) applied to those rows; hence, b_i+1 is in column 1 or 2 by Lemma 4.2(ii).

Case $a(i,0) = b_i+1$. By a similar argument b_i is not in column 0; hence b_i is in column 1 or 2 (Table 3).

This proves the inductive step for type A.

Types B and C. The reasoning is like that for Type A, but more complicated. Define u_i, v_i, w_i as in equations (4.3)–(4.5). Then $u_i, v_i, w_i < b_i$ so by (i) each of u_i, v_i, w_i occurs exactly once in rows $0, \dots, i-1$. By 4.3(i) $\mathrm{S}_1(u_i) = b_i$ or b_i+1, $\mathrm{S}_1(v_i) = b_i+1$ or b_i+2, and $\mathrm{S}_3(w_i) = b_i$ or b_i+2.

Hence, at least two of b_i, b_i+1, b_i+2 occur in rows $0, \dots, i-1$. Suppose, for a contradiction, that all three occur. By (ii) none is in column 0. Hence by Proposition 4.3(ii) either u_i or v_i occurs twice in rows $0, \dots, i-1$, contrary to (i). Thus exactly two of b_i, b_i+1, b_i+2 occur in rows $0, \dots, i-1$, and the third is chosen as $a(i,0)$.

For (iii), there are three cases.

Case $a(i,0) = b_i$. As in type A, b_i+1 is in column 1 or 2. Since $\mathrm{S}_3(w_i) \neq b_i$, we have $\mathrm{S}_3(w_i) = b_i+2$, so b_i+2 is not in column 1 or 2.

Case $a(i,0) = b_i+1$. Since $\mathrm{S}_3(w_i) = b_i$ or b_i+2, one of these is not is column 1 or 2. Suppose, for a contradiction, that neither is. Then b_i occurs as $\langle 4, f, \dots\rangle$ (Type B), resp. $\langle e, \dots\rangle$ (Type C), and b_i+2 occurs as $\langle 5, f, \dots\rangle$ or $\langle d, \dots\rangle$ (Type B), resp. $\langle 3, e, \dots\rangle$ (Type C). Then $w_i = \mathrm{S}_{-3}(b_i) = \mathrm{S}_{-3}(b_i+2)$, so that w_i occurs twice in rows $0, \dots, i-1$, contrary to (i).

Case $a(i,0) = b_i+2$. As in block Type A, b_i+1 is in column 1 or 2. Since $\mathrm{S}_3(w_i) \neq b_i+2$, we have $\mathrm{S}_3(w_i) = b_i$, so b_i is not in column 1 or 2.

This proves the inductive step for Types B and C. □

The following is an alternative proof of a known result.

Proposition 4.5. *Suppose a Stolarsky interspersion is generated in terms of QZReps, as described above. Then it contains every positive integer exactly once.*

Proof. Let n be a positive integer, and let n belong to block i. By Proposition 4.4(i) n occurs exactly once in rows $0, \ldots, i$, and by Proposition 4.4(ii) all terms in succeeding rows are greater than n. □

5. The "Even Second Column" Array

Proposition 5.1. *Suppose the ESC array is generated in terms of QZReps, as described above. For integer $i \geq 0$:*

(i) The QZRep used for $a(i, 0)$ has even T_2.
(ii) (a) If block i is of type B then $a(i, 0) = b_i$ or $b_i + 1$, and (b) if also i is odd then $a(i, 0) = b_i + 1$.
(iii) (a) If block i is of type C then $a(i, 0) = b_i + 1$ or $b_i + 2$, and (b) if also i is even then $a(i, 0) = b_i + 1$.
(iv) The QZRep used for $a(i, 0)$ does not begin $\langle 1, 3, \ldots\rangle$.
(v) The QZRep used for $a(i, 0)$ does not begin $\langle 0, d, \ldots\rangle$ with $d > 3$.

Proof. By construction, $a(i, 0)$ is written with its odd or even QZRep according as i is odd or even, so (i) follows from Proposition 4.1.

For the rest, use induction on i. The result is easily checked for $i = 0, 1$. Suppose that $i > 1$ and (ii)–(v) hold with $i - 1$ in place of i. The inductive step depends on the type of block i (refer to Table 3).

Type A. Here (ii) and (iii) are vacuously true, and all four QZReps for $b_i, b_i + 1$ satisfy (iv) and (v).

Type B. For (ii)(a) we must show that $b_i + 2$ occurs in rows $0, \ldots, i - 1$. Define v_i as in equation (4.4). Then $v_i < b_i$, so by Proposition 4.4(i) v_i occurs in rows $0, \ldots, i - 1$, and not in its odd QZRep $\langle 1, 3, \ldots\rangle$ since by Lemma 4.2(i) that would be the first term of a row, contrary to (iv). Hence, v_i occurs in its even QZRep and $b_i + 2 = \mathrm{S}_1(v_i)$.

For (ii)(b) we assume that i is odd and show that b_i occurs in rows $0, \ldots, i - 1$. Define

$$s = \mathrm{val}\langle 1, f - 2, \ldots\rangle = \mathrm{val}\langle 2, f - 2, \ldots\rangle, \quad t = \mathrm{val}\langle 3, f - 1, \ldots\rangle = \mathrm{val}\langle 0, 3, f - 1, \ldots\rangle.$$

By Proposition 4.4(i) s and t occur in rows $0, \ldots, i - 1$. If s occurs as $\langle 2, f - 2, \ldots\rangle$ then $b_i = \mathrm{S}_2(s)$ and we are done. Suppose then that s occurs as $\langle 1, f - 2, \ldots\rangle$.

Clearly s is in column 0 or 1. If s is in column 1 then $\mathrm{S}_{-1}(s) = \langle 0, f - 3, \ldots\rangle$, so by (i) $\mathrm{T}_2\langle 0, f - 3, \ldots\rangle$ is even. But $\mathrm{T}_2\langle 0, f - 3, \ldots\rangle = \mathrm{T}_2\langle 4, f, \ldots\rangle$, which is odd by Proposition 4.1 since $\mathrm{val}\langle 4, f, \ldots\rangle = b_i$ is in an odd block by hypothesis. Hence s is in column 0, and by (i) $\mathrm{T}_2\langle 1, f - 2, \ldots\rangle$ is even. By Proposition 3.3(iii),

$$\mathrm{T}_2\langle 0, 3, f - 1, \ldots\rangle = 2 + \mathrm{T}_2\langle 2, f - 1, \ldots\rangle \equiv \mathrm{T}_2\langle 0, f - 3, \ldots\rangle + \mathrm{T}_2\langle 1, f - 2, \ldots\rangle \pmod 2.$$

The RHS terms are respectively odd and even as just shown; hence by (i) $\langle 0, 3, f - 1, \ldots\rangle$ cannot start a row and so cannot occur at all. Hence, t occurs as $\langle 3, f - 1, \ldots\rangle$, and $b_i = \mathrm{S}_1(t)$.

For type B, (iii) is vacuously true, all four QZReps for b_i, $b_i + 1$ satisfy (v), and (iv) holds because (ii)(b) ensures that the QZRep $\langle 1, 3, f, \ldots\rangle$ is never chosen.

Type C. For (iii)(a) we must show that b_i occurs in rows $0, \ldots, i - 1$. Define u_i as in equation (4.3). Then $u_i < b_i$, so by Proposition 4.4(i) u_i occurs in rows $0, \ldots, i - 1$, and not in its even QZRep $\langle 0, e-1, \ldots\rangle$ since that would be the first term of a row and contradict (v). Hence, u_i occurs in its odd QZRep, and $b_i = \mathrm{S}_1(u_i)$.

For (iii)(b) we assume that i is even and show that $b_i + 2$ occurs in rows $0, \ldots, i-1$. Define

$$s = \mathrm{val}\langle 1, e-2, \ldots\rangle = \mathrm{val}\langle 2, e-2, \ldots\rangle, \quad t = \mathrm{val}\langle 1, e-1, \ldots\rangle = \mathrm{val}\langle 2, e-1, \ldots\rangle.$$

By Proposition 4.4(i) s and t occur in rows $0, \ldots, i-1$. If s occurs as $\langle 1, e-2, \ldots\rangle$ then $b_i + 2 = \mathrm{S}_2(s)$ and we are done. Suppose then that s occurs as $\langle 2, e-2, \ldots\rangle$.

By Lemma 4.2(i), s is in column 0 or 1. If s is in column 1 then $\mathrm{S}_{-1}(s) = \langle 1, e-3, \ldots\rangle$, so by (i) $\mathrm{T}_2\langle 1, e-3, \ldots\rangle$ is even. But $\mathrm{T}_2\langle 1, e-3, \ldots\rangle = \mathrm{T}_2\langle 3, e, \ldots\rangle$, which is odd by Proposition 4.1 since $\mathrm{val}\langle 3, e\rangle = b_i + 2$ is in an even block by hypothesis. Hence, s is in column 0, and by (i) $\mathrm{T}_2\langle 2, e-2, \ldots\rangle$ is even. By Proposition 3.3(iii),

$$\mathrm{T}_2\langle 1, e-1, \ldots\rangle = \mathrm{T}_2\langle 3, e-1, \ldots\rangle \equiv \mathrm{T}_2\langle 1, e-3, \ldots\rangle + \mathrm{T}_2\langle 2, e-2, \ldots\rangle \pmod 2.$$

The RHS terms are respectively odd and even, as just shown; hence by (i) $\langle 1, e-1, \ldots\rangle$ cannot start a row and so cannot occur at all (Lemma 4.2(i)). Hence, t occurs as $\langle 2, e-1, \ldots\rangle$, and $b_i + 2 = \mathrm{S}_1(t)$.

For type C, (ii) is vacuously true, and all four QZReps for $b_i + 1$, $b_i + 2$ satisfy (iv) and (v). □

Theorem 5.2. *The second term in every row of the ESC array is even.*

Proof. The result follows from the construction of the array in terms of QZReps, and Propositions 5.1(i), 3.3(ii). □

6. The "Even First Column" Array

The "even first column" (EFC) array is defined by taking $\delta_i = 0$ for even $i > 0$, 1 for odd i. Kimberling [1] proved that column 0 of this array is all even apart from the initial 1. The techniques above can be applied to give an alternative proof, as follows.

Proposition 6.1. *Suppose the EFC array is generated in terms of QZReps, as described above. For integer $i \geq 0$:*

(i) If $i > 0$ then the QZRep used for $a(i, 0)$ has odd T_2.
(ii) If block i is of type B or C, and i is even, then $a(i, 0) = b_i$ or $b_i + 1$.
(iii) If block i is of type B or C, and i is odd, then $a(i, 0) = b_i + 1$ or $b_i + 2$.
(iv) If $i > 0$ then $a(i, 0)$ is even.

Proof. By construction, we choose the even QZRep of $a(i, 0)$ for odd i, and vice versa; so (i) follows from Proposition 4.1.

For the rest, use induction on i. The result is easily checked for $i = 0, 1$. Suppose that $i > 1$ and (ii)–(iv) hold with $i - 1$ in place of i. Then by (i) and Proposition 3.3(ii) $a(k, 1)$ is odd for $1 \leq k \leq i - 1$, so in rows $1, \ldots, i-1$ the parity follows the pattern EOOEOO.... In row 0 the pattern is OEOOEO....

For the inductive step in (ii) and (iii) refer to equations (4.3)–(4.4). Let Q be the odd QZRep of v_i if i is even, resp., the even QZRep of u_i if i is odd. There are 5 possibilities for Q:

$$\langle 1, 3, f-1, \ldots\rangle, \quad \langle 1, 3, 5, \ldots, d-2, \ldots\rangle, \quad \langle 1, e-1, \ldots\rangle, \quad \langle 0, 3, f-1, \ldots\rangle, \quad \langle 0, e-1, \ldots\rangle.$$

Increment each term in Q by 1 to give a QZRep Q^+; then clearly $\mathrm{T}_1 Q = \mathrm{T}_2 Q^+$ and Q^+ has opposite parity to Q; hence by Proposition 4.1, $\mathrm{T}_1 Q$ is even.

Define $t = \text{val}Q$. Then $t < b_i$ and hence, t occurs in rows $0, \ldots, i-1$. Suppose, for a contradiction, that t occurs in the QZRep Q. In each of the 5 cases, t starts a row (Lemma 4.2(i), or obvious), and not row 0 by Lemma 4.2(iii); hence by (iv) t is even, and by Proposition 3.3(i) $\mathrm{T}_0 Q$ is even. Since $\mathrm{T}_1 Q$ is even (as shown), Proposition 3.3(iii) implies $\mathrm{T}_2 Q$ is even, contrary to (i). Hence, t occurs in its other QZRep in (4.3)–(4.4), i.e. the even QZRep of v_i if i is even, resp., the odd QZRep of u_i if i is odd. Proposition 4.3(i) then implies $\mathrm{S}_1(t) = b_i + 2$ or b_i. Thus, if i is even (resp. odd) then $b_i + 2$ (resp. b_i) occurs before row i, leaving the possibilities for $a(i, 0)$ stated in (ii) (resp. (iii)).

For (iv) the inductive step depends on the type of block i (refer to Table 3).

Type A. If $a(i, 0) = b_i$ then $b_i + 1$ occurs in rows $1, \ldots, i-1$, and in column 1 or 2 (Lemma 4.2(iv),(ii)). Hence, $b_i + 1$ is odd. Hence, $a(i, 0)$ is even. If $a(i, 0) = b_i + 1$ then b_i occurs in rows $1, \ldots, i-1$ (Lemma 4.2(iv)) and in column 1 or 2 (Table 3) and again $a(i, 0)$ is even.

Types B and C. If $a(i, 0) \neq b_i + 1$ then as in type A $b_i + 1$ is odd, hence, $a(i, 0)$ is even.

If $a(i, 0) = b_i + 1$ then neither b_i nor $b_i + 2$ can start a row. Hence for type B, b_i occurs as $\langle 4, f, \ldots \rangle$, and $b_i + 2$ occurs as $\langle 5, f, \ldots \rangle$ or $\langle d, \ldots \rangle$; for type C, b_i occurs as $\langle e, \ldots \rangle$ and $b_i + 2$ occurs as $\langle 3, e, \ldots \rangle$. For even (resp. odd) i the QZRep of b_i (resp. $b_i + 2$) has even T_2 by Proposition 4.1. Hence by Proposition 3.3(ii), $\mathrm{S}_1(b_i)$ resp., $\mathrm{S}_1(b_i + 2)$ is even. Hence, b_i resp. $b_i + 2$ is odd, and so $a(i, 0) = b_i + 1$ is even. □

7. Wythoff, Dual Wythoff, and Original Stolarsky Arrays

This section discusses the class of Stolarsky interspersions defined by

$$a(i, 1) = \lfloor \Phi a(i, 0) + \lambda \rfloor \quad (i \geq 1), \tag{7.1}$$

where $0 \leq \lambda \leq 1$ and λ is constant for a particular array. Here δ_i $(i \geq 1)$ is determined implicitly by (7.1). In general there seems to be no simple formula giving δ_i in terms of i.

If $\lambda = 1$ we get the Wythoff array [2], with $\delta_i = 1$ for all i. If $\lambda = 0$ we get the dual of the Wythoff array [2], with $\delta_i = 0$ for all $i > 0$. If $\lambda = 1/2$ we get Stolarsky's original array [4], with no known simple formula for δ_i.

It will be shown below that the value of λ makes less difference than one might expect. In particular, the Wythoff array is generated for all $\lambda \in [\Phi^{-1}, 1]$ and the Wythoff dual for all $\lambda \in [0, \Phi^{-2}]$.

Lemma 7.1. *(after Stolarsky). Let x be a positive integer, $0 \leq \lambda \leq 1$, and $y = \lfloor \Phi x + \lambda \rfloor$. Then*

(i) $x = \lfloor \Phi^{-1} y + 1 - \lambda \rfloor$.
(ii) $x + y = \lfloor \Phi y + 1 - \lambda \rfloor$.

Proof. Write

$$\Phi x + \lambda = y + \eta, \tag{7.2}$$

so that $0 \leq \eta < 1$. Since also $0 \leq \lambda \leq 1$ we have

$$1 > (1 - \Phi^{-1})\lambda + \Phi^{-1}\eta \geq 0. \tag{7.3}$$

The equality holds in (7.3) if and only if $\lambda = \eta = 0$, but in that case (7.2) implies $\Phi x = y$, which is impossible since Φ is irrational. Hence, $\geq$ can be sharpened to $>$. Since from (7.2) $x = \Phi^{-1}(y + \eta - \lambda)$, subtracting (7.3) from $x + 1$ gives

$$x < \Phi^{-1} y + 1 - \lambda < x + 1,$$

and (i) follows. (ii) then follows from $\Phi^{-1} + 1 = \Phi$. □

Proposition 7.2. *In the array generated by* (7.1)*, we have for* $i, j \geq 1$

$$a(i,j) = \begin{cases} \lfloor \Phi a(i,j-1) + \lambda \rfloor & \text{if } j \text{ is odd} \\ \lfloor \Phi a(i,j-1) + 1 - \lambda \rfloor & \text{if } j \text{ is even.} \end{cases} \tag{7.4}$$

Proof. By induction on j. The case $j = 1$ is the definition (7.1). The inductive step follows from Lemma 7.1(ii) and the Fibonacci-type relation that defines the row. $\square$

Stolarsky (Lemma 2 of [4]) proved Lemma 7.1(ii) for $\lambda = 1/2$. In this case we have

$$a(i,j) = \lfloor \Phi a(i,j-1) + 1/2 \rfloor \quad (i, j \geq 1). \tag{7.5}$$

In fact Stolarsky used (7.5) to *define* his array, and derived the Fibonacci-type relation as a consequence. If in (7.5) 1/2 is replaced by general λ, we get an alternative generalization of Stolarsky's array, in which the Fibonacci relation does not necessarily hold; this will not be discussed here.

Lemma 7.3. *Let x be a positive integer, $0 \leq \lambda \leq 1$, and $y = \lfloor \Phi x + \lambda \rfloor$. Let $\langle c_0, c_1, \ldots, c_q \rangle$ be the ZRep of x.*

(i) If c_0 is even and $\lambda \geq \Phi^{1-c_0}$ then $y = \lfloor \Phi x \rfloor + 1$.
(ii) If $c_0 = 2$, c_1 is even, and $\lambda \leq \Phi^{-2}$ then $y = \lfloor \Phi x \rfloor$.
(iii) If c_0 is odd and $\lambda \leq 1 - \Phi^{1-c_0}$ then $y = \lfloor \Phi x \rfloor$.

Proof. Since $x = F_{c_0} + F_{c_1} + \cdots + F_{c_q}$, Lemma 2.2(i) implies $\Phi x = n - \xi$, where

$$n = F_{c_0+1} + F_{c_1+1} + \cdots + F_{c_q+1}, \quad \xi = (-\Phi)^{-c_0} + (-\Phi)^{-c_1} + \cdots + (-\Phi)^{-c_q}.$$

Lemma 2.3 with $p = 0$ gives $\xi = (-\Phi)^{-c_0}\theta$, where $\Phi^{-1} < \theta < \Phi$.

For (i), $\Phi x = n - |\xi|$, where $|\xi| < \Phi^{1-c_0} \leq \lambda$. Hence, $y = \lfloor \Phi x + \lambda \rfloor = n = \lfloor \Phi x \rfloor + 1$.

For (ii), Lemma 2.3 with $p = 1$ gives $|\xi| = \Phi^{-2} + \Phi^{-c_1}\theta$ where $\Phi^{-1} < \theta < \Phi$. Hence, $\Phi x = n - |\xi|$ where $|\xi| > \Phi^{-2} \geq \lambda$. Hence, $y = \lfloor \Phi x + \lambda \rfloor = n - 1 = \lfloor \Phi x \rfloor$.

For (iii), $\Phi x = n + |\xi|$ with $|\xi| < \Phi^{1-c_0} \leq 1 - \lambda$. Hence, $y = \lfloor \Phi x + \lambda \rfloor = n = \lfloor \Phi x \rfloor$. $\square$

The Wythoff Array. The Wythoff array is generated by taking $\lambda = 1$, or equivalently $\delta_i = 1$ for all i. This array was studied by Kimberling [2], who showed that column j consists of exactly those integers whose ZRep has lowest index $j + 2$. The following proposition proves Kimberling's result using the concepts of this paper.

Proposition 7.4. *In the array generated by* (7.1) *with* $\Phi^{-1} \leq \lambda \leq 1$*:*

(i) $\delta_i = 1$ for all i.
(ii) $a(i,0) = b_i$ if block i is of Type A, and $a(i,0) = b_i + 1$ otherwise.
(iii) The QZRep used for $a(i,0)$ starts $\langle 2, \ldots \rangle$, and all such QZReps appear in column 0.

Proof. By induction over i. The result is clear for $i = 0$. For $i = 1$, note that $a(1,0) = 4$ and

$$6 < 4\Phi < 7 < 4\Phi + \Phi^{-1} < 4\Phi + 1 < 8.$$

Hence, $a(1,1) = \lfloor 4\Phi + \lambda \rfloor = 7$, $\delta_1 = 1$, and $a(1,0)$ appears in the even QZRep $\langle 2, 4 \rangle$. Now suppose $i > 1$ and the result holds for $0, \ldots, i-1$. The inductive step depends on the type of block i. Again d, e, f are as in Table 3.

Type A. Let $u = \mathrm{val}\langle 2, d-1 \rangle$. Then u occurs in rows $0, \ldots, i-1$ in the QZRep $\langle 2, d-1, \ldots \rangle$, hence, $\mathrm{S}_1(u) = b_i + 1$. Hence, $a(i,0) = b_i$ and its ZRep is $\langle 2, d, \ldots \rangle$.

Type B. Let $u = \mathrm{val}\langle 2, f-2, \ldots\rangle$. Then u occurs in rows $0, \ldots, i-1$ in the QZRep $\langle 2, f-2, \ldots\rangle$, hence, $b_i = \mathrm{S}_1(u)$. Similarly,

$$b_i + 2 = \begin{cases} \mathrm{S}_1(v), & v = \mathrm{val}\langle 2, f-3, \ldots\rangle & \text{if } f > 6 \\ \mathrm{S}_{d-2}(v), & v = \mathrm{val}\langle 2, \ldots\rangle & \text{if } f = 6. \end{cases}$$

Hence, $a(i,0) = b_i + 1$ and its ZRep is $\langle 2, 4, f, \ldots\rangle$.

Type C. Here $b_i = \mathrm{S}_{e-2}(u)$ where $u = \mathrm{val}\langle 2, \ldots\rangle$, and $b_i + 2 = \mathrm{S}_1(v)$ where $v = \mathrm{val}\langle 2, e-1, \ldots\rangle$. Hence, $a(i,0) = b_i + 1$ and its ZRep is $\langle 2, e, \ldots\rangle$.

Thus for all block types the ZRep of $a(i,0)$ begins $\langle 2, \ldots\rangle$, so Lemma 7.3(i) implies $a(i,1) = \lfloor \Phi a(i,0)\rfloor + 1$, i.e. $\delta_i = 1$. Hence the even QZRep $\langle 2, \ldots\rangle$ is used for $a(i,0)$.

Table 3 shows that each block contains exactly one QZRep starting with $\langle 2, \ldots\rangle$, and the above shows that for each block this QZRep is always the one chosen for $a(i,0)$. Hence all such QZReps appear in column 0. □

The Dual of the Wythoff Array. The Wythoff dual is generated by taking $\lambda = 0$, or equivalently $\delta_i = 0$ for all $i > 0$.

Proposition 7.5. *In the array generated by* (7.1) *with* $0 \le \lambda \le \Phi^{-2}$, *we have for all* $i > 0$*:*

(i) $\delta_i = 0$.
(ii) $a(i,0) = b_i + 1$.
(iii) The QZRep used for $a(i,0)$ *starts either* $\langle 1, e, \ldots\rangle$ *(e even) or* $\langle 3, d, \ldots\rangle$ *(d odd), and all such QZReps appear in column 0.*

Proof. By induction over i. For $i = 1$, note that $a(1,0) = 4$ and

$$6 < 4\Phi < 4\Phi + \Phi^{-2} < 7.$$

Hence, $a(1,1) = \lfloor 4\Phi + \lambda\rfloor = 6$, $\delta_1 = 0$, and $a(1,0)$ appears in the odd QZRep $\langle 1, 4\rangle$.

Now suppose $i > 1$ and the result holds for $1, \ldots, i-1$. The inductive step depends on the type of block i. Again d, e, f are as in Table 3. Note that by (iii) rows $0, \ldots, i-1$ do not contain a QZRep $\langle 0, \ldots\rangle$ or $\langle 2, h, \ldots\rangle$ where h is even.

Type A. Let $u = \mathrm{val}\langle 1, d-1\rangle$. Then u occurs in rows $0, \ldots, i-1$ in the QZRep $\langle 1, d-1, \ldots\rangle$ not $\langle 2, d-1, \ldots\rangle$, hence, $\mathrm{S}_1(u) = b_i$. Hence, $a(i,0) = b_i + 1$ and its ZRep is $\langle 3, d, \ldots\rangle$. Since $\lambda \le \Phi^{-2}$, we have *a fortiori* $\lambda \le 1 - \Phi^{-2}$, and Lemma 7.3(iii) implies $a(i,1) = \lfloor \Phi a(i,0)\rfloor$, i.e. $\delta = 0$. Hence the odd QZRep $\langle 3, d, \ldots\rangle$ is used for $a(i,0)$.

Type B. Here

$$b_i = \begin{cases} \mathrm{S}_1(u), & u = \mathrm{val}\langle 3, f-1, \ldots\rangle & \text{if } f \text{ is even} \\ \mathrm{S}_3(u), & u = \mathrm{val}\langle 1, f-3, \ldots\rangle & \text{if } f \text{ is odd.} \end{cases}$$

If $f > 6$ then

$$b_i + 2 = \begin{cases} \mathrm{S}_4(v), & v = \mathrm{val}\langle 1, f-4, \ldots\rangle & \text{if } f \text{ is even} \\ \mathrm{S}_2(v), & v = \mathrm{val}\langle 3, f-2, \ldots\rangle & \text{if } f \text{ is odd.} \end{cases}$$

If $f = 6$ let $b_i + 2 = \mathrm{val}\langle d, g, \ldots\rangle$ with odd $d \ge 7$. Then

$$b_i + 2 = \begin{cases} \mathrm{S}_{d-1}(v), & v = \mathrm{val}\langle 1, g-d+1, \ldots\rangle & \text{if } g \text{ is even} \\ \mathrm{S}_{d-3}(v), & v = \mathrm{val}\langle 3, g-d+3, \ldots\rangle & \text{if } g \text{ is odd.} \end{cases}$$

Hence, $a(i,0) = b_i + 1$ and its ZRep is $\langle 2, 4, f, \ldots\rangle$. Hence Lemma 7.3(ii) implies $a(i,1) = [\Phi a(i,0)]$, i.e. $\delta = 0$. Hence the odd QZRep $\langle 1, 4, f, \ldots\rangle$ is used for $a(i,0)$.

Type C. Let $b_i = \mathrm{val}\langle e, g, \dots\rangle$ (even $e \geq 6$). Then

$$b_i = \begin{cases} \mathrm{S}_{e-3}(u), & u = \mathrm{val}\langle 3, g-e+3, \dots\rangle \qquad \text{if } g \text{ is even} \\ \mathrm{S}_{e-1}(u), & u = \mathrm{val}\langle 1, g-e+1, \dots\rangle \qquad \text{if } g \text{ is odd.} \end{cases}$$

Also $b_i + 2 = \mathrm{S}_2(v)$ where $v = \mathrm{val}\langle 1, e-2, \dots\rangle$.

Hence, $a(i,0) = b_i + 1$ and its ZRep is $\langle 2, e, \dots\rangle$. Lemma 7.3(ii) implies $a(i,1) = \lfloor \Phi a(i,0)\rfloor$, i.e. $\delta = 0$. Hence the odd QZRep $\langle 1, e, \dots\rangle$ is used for $a(i,0)$.

Table 3 shows that every block contains exactly one QZRep of the type stated, and the above shows that for each block this QZRep is always the one chosen for $a(i,0)$. Hence all such QZReps appear in column 0. □

Proposition 7.6. *In the array generated by* (7.1) *with* $0 \leq \lambda \leq 1$, *we have* $a(i,0) = b_i + 1$ *whenever block* i *is of type B or C.*

Proof. For $\lambda \in [\Phi^{-1}, 1]$ (resp. $[0, \Phi^{-2}]$) the result is already proved in Proposition 7.4 (resp. Proposition 7.5). So assume that $\Phi^{-2} < \lambda < \Phi^{-1}$. Let block i be of type B or C (hence $i > 0$) and suppose, for a contradiction, that $b_i + 1$ does not begin a row. Define $y = b_i + 1$ and $x = \mathrm{S}_{-1}(y)$. By Proposition 7.2, $y = \lfloor \Phi x + \lambda'\rfloor$, where $\lambda' = \lambda$ or $1 - \lambda$ and in either case $\Phi^{-2} < \lambda' < \Phi^{-1}$ (since $\Phi^{-1} + \Phi^{-2} = 1$). By Lemma 7.1(i), $x = \lfloor \Phi^{-1} y + 1 - \lambda'\rfloor$.

Table 3 shows that the ZRep of y $(= b_i + 1)$ begins $\langle 2, e, \dots\rangle$ with e even. Hence by Lemma 2.2(ii)

$$\Phi^{-1} y + 1 - \lambda' = (F_1 + F_{e-1} + \cdots) + \mu, \quad \text{where } \mu = 1 - \lambda' - \Phi^{-2} - \Phi^{-e} \pm \cdots .$$

Lemma 2.3 with $p = 0$ gives $\mu < 1 - \lambda' - \Phi^{-3} < 1$ and $\mu > 1 - \lambda' - \Phi^{-1} > -1$. Hence

$$x = \lfloor \Phi^{-1} y + 1 - \lambda'\rfloor = (F_1 + F_{e-1} + \cdots) - \kappa,$$

where $\kappa = 0$ if $\mu \geq 0$ or $\kappa = 1$ if $\mu < 0$. By Lemma 2.2(i) and Lemma 2.3,

$$\begin{aligned} \Phi x + \lambda' &= (F_2 + F_e + \cdots) + \Phi^{-1} + \Phi^{1-e}\theta - \Phi\kappa + \lambda' \quad (\text{where } \Phi^{-1} < \theta < \Phi) \\ &= y + \lambda' - \Phi\kappa + \Phi^{-1} + \Phi^{1-e}\theta. \end{aligned}$$

If $\mu \geq 0$ this gives

$$\Phi x + \lambda' > y + \lambda' + \Phi^{-1} + \Phi^{-e} > y + \Phi^{-2} + \Phi^{-1} + 0 = y + 1,$$

contradicting $\lfloor \Phi x + \lambda'\rfloor = y$. If $\mu < 0$ then

$$\Phi x + \lambda' < y + \lambda' - \Phi + \Phi^{-1} + \Phi^{2-e} \leq y + \Phi^{-1} - \Phi + \Phi^{-1} + \Phi^{-2} = y,$$

again contradicting $\lfloor \Phi x + \lambda'\rfloor = y$. □

References

[1] C. Kimberling, *The first column of an interspersion*, The Fibonacci Quarterly, **32.3** (1994), 301–315.
[2] C. Kimberling, *The Zeckendorf array equals the Wythoff array*, The Fibonacci Quarterly, **33.1** (1995), 3–8.
[3] C. Kimberling, *Edouard Zeckendorf*, The Fibonacci Quarterly, **36.4** (1998), 416–418.
[4] K. B. Stolarsky, *A set of generalized Fibonacci sequences such that each natural number belongs to exactly one*, The Fibonacci Quarterly, **15.2** (1977), 224.
[5] E. Zeckendorf, *Représentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bulletin de la Société Royale des Sciences de Liège, **41** (1972), 179–182.

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