EDGE-LENGTH RATIOS BETWEEN DUAL PLATONIC SOLIDS: A SURPRISINGLY NEW RESULT INVOLVING THE GOLDEN RATIO

STEPHEN R. WASSELL AND SAMANTHA BENITO

ABSTRACT. A thorough review of the appropriate literature reveals the possibility of a fundamental ratio going unnoticed until now, or at least being absent from the literature: the edge-length ratio between a regular dodecahedron and its circumscribing dual (polar reciprocal) icosahedron when paired vertex to face, namely one third of the golden ratio. This ratio completes an elegant triplet of ratios for vertex-to-face dual pairings when the outer Platonic solid is the tetrahedron, octahedron, and icosahedron (i.e., those with triangular faces), specifically $1 : 3$, $\sqrt{2} : 3$, and $\phi : 3$, respectively.

1. A Substantial Gap in the Literature

As improbable as it may seem, a fundamental ratio concerning the regular icosahedron and its dual dodecahedron appears to be previously unknown, despite the fact that the ratio in question is $\phi : 3$, where ϕ denotes the golden ratio, one of the most celebrated numbers of all time. We are referring to two of the five Platonic solids, the well-known subjects of the thirteenth and final book of Euclid's *Elements*, as well as to the extreme and mean ratio (the ancient Greek term for what is now usually called the golden ratio), which appears in several places in Euclid's famous text.[1] For that matter, the analogous ratio involving the regular octahedron and cube (hexahedron) also appears to be absent from the literature, and it is similarly simple, namely $\sqrt{2} : 3$.

It is important to be specific regarding what we mean by dual, since there are various concepts of duality in geometry, topology, etc. The basic fact is that four of the five Platonic solids pair off in duals, also called reciprocals (more precisely, polar reciprocals, as we discuss momentarily), wherein the cube and octahedron are duals of each other, as are the dodecahedron and icosahedron, whereas the tetrahedron is its own dual. We shall consider two different pairings, vertex-to-face and edge-to-edge.

A vertex-to-face pairing is the result of undoubtedly the easiest method for finding the dual of any regular polyhedron: simply connect the centroids of its adjacent faces with line segments. Figures 1(a) and 1(b) show the possibilities for the cube and octahedron. Notice that there are two different configurations for the vertex-to-face pairings, depending on which is the outer polyhedron and which is the inner one.

[1] Appearances include Books II, IV, VI, X, and XIII; Euclid explicitly defines the extreme and mean ratio in Book VI, Definition 3 [8, vol. 2, p. 188], but already in Book II, Proposition 11 [8, vol. 1, p. 402] he provides a construction that divides a given line segment in extreme and mean ratio. The golden ratio is also known as the golden section, golden number, golden mean, and divine proportion, among other monikers, and the name is sometimes capitalized. Regardless of what it is called, it is the number $(1 + \sqrt{5})/2$, approximately 1.618. The golden ratio is denoted by τ instead of ϕ in some sources.

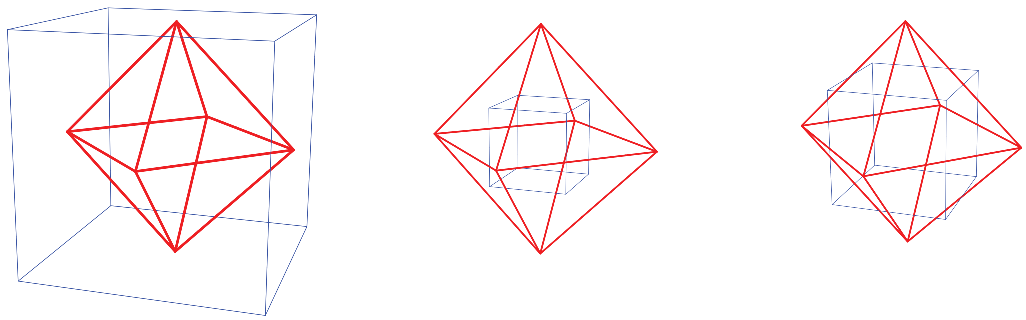

FIGURE 1. Three dual pairings for the cube and octahedron: (a) vertex-to-face with cube enclosing octahedron; (b) vertex-to-face with octahedron enclosing cube; (c) edge-to-edge.

In order to motivate edge-to-edge pairings, let us consider polar reciprocation at least briefly. Polar reciprocation maps points to planes (and vice versa) about a given sphere; in the present context, the points are the vertices of a Platonic solid, the planes correspond to the faces of its dual, and the sphere could be the solid's circumscribing sphere (the sphere passing through its vertices), its inscribing sphere (the sphere tangent to its faces), or its midsphere (the sphere tangent to its edges).[2] Using either the circumscribing sphere or the inscribing sphere results in a vertex-to-face pairing, while an edge-to-edge pairing is obtained by taking the polar reciprocal about the midsphere. The upshot in this last case is that the polyhedron and its dual are superimposed with corresponding edges serving as perpendicular bisectors of each other. Figure 1(c) shows the edge-to-edge dual relationship for the cube and octahedron. Figure 2 shows the vertex-to-face and edge-to-edge pairings for the dodecahedron and icosahedron, and Figure 3 shows those for two tetrahedra. Turning to the edge-length ratios of pairs of duals, meaning simply the ratio of the edge length of one to that of the other, it should be clear from Figures 1 through 3 that with the edge-to-edge pairings there are three ratios to consider, whereas with the vertex-to-face pairings there are five ratios.

2. Edge-to-Edge Dual Pairings

The three ratios for the edge-to-edge pairings are well documented in the literature, as we discuss in depth below. For the self-dual tetrahedron, the ratio is, of course, $1 : 1$; the ratio is $1 : \sqrt{2}$ for the cube and octahedron; and it is $1 : \phi$ for the dodecahedron and icosahedron. Fans of the math/art interface will surely appreciate the elegant sequence given by these edge-length ratios for the edge-to-edge pairings, since these three ratios are crucial in many geometric constructions found in art. It is beyond the scope of this article to describe the many ways in which they are crucial, but we can motivate a basic level of appreciation by considering the three rectangles associated with these ratios.

The rectangle associated with the ratio $1 : 1$ is, of course, the square, whose importance is self-evident. The $\sqrt{2}$ rectangle, i.e., the rectangle whose length-to-width ratio is equal to $1 : \sqrt{2}$, is the unique rectangle having the following property: two such rectangles placed side by side together form a larger rectangle with the same proportion. The importance of this

[2]For a more detailed and nicely illustrated description of polar reciprocation, see [16, p. 1–5]; it is also easy to find some decent discussions on the Internet.

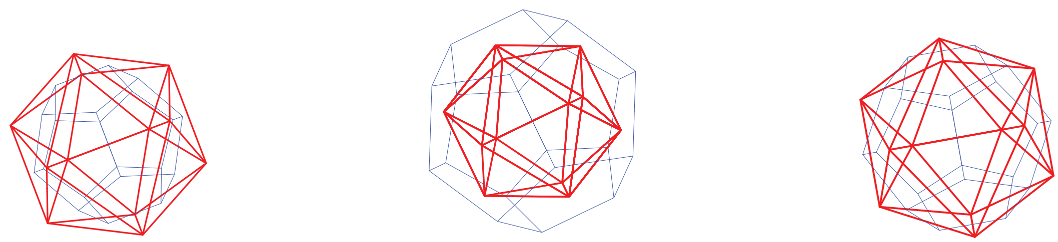

FIGURE 2. Three dual pairings for the dodecahedron and icosahedron: (a) vertex-to-face with icosahedron enclosing dodecahedron; (b) vertex-to-face with dodecahedron enclosing icosahedron; (c) edge-to-edge.

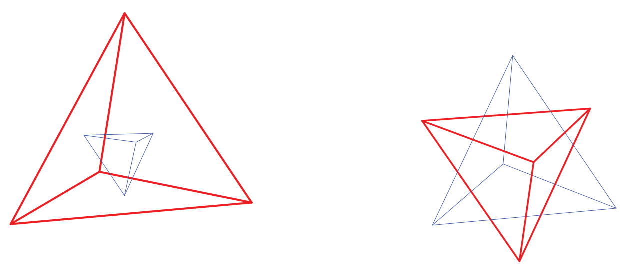

FIGURE 3. Two dual pairings for the self-dual tetrahedron: (a) vertex-to-face; (b) edge-to-edge.

property can be appreciated, at least briefly, by realizing that there are many examples in art and architecture of placing rectangles side by side—and by realizing that repetition of ratios can imbue a given design with pleasing aesthetic properties.[3] More generally, and using a somewhat technical term, any *ad quadratum* geometric construction depends inherently on the square root of two; perhaps the most basic and well-known example is drawing the diagonal of a square.

Finally, the golden ratio rectangle, or golden rectangle, i.e., the rectangle whose length-to-width ratio is equal to $1 : \phi$, is the unique rectangle having an only slightly different property:

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[3]It may be of interest in this regard that the $\sqrt{2}$ rectangle is the basis for the "international standard" (ISO) paper system, used in Europe and elsewhere. Letter-size paper in the U.S. measures 8.5 by 11 inches, whereas the closest ISO paper size, called A4, measures 210 by 297 millimeters (approximately 8.3 by 11.7 inches), which is, up to the nearest millimeter, a $\sqrt{2}$ rectangle. A sheet of A4 paper cut or folded in half lengthwise produces A5 paper size, which is also a $\sqrt{2}$ rectangle. In the other direction, two sheets of A4 paper joined together (long sides adjacent) produce A3 paper, again a $\sqrt{2}$ rectangle, roughly equivalent to the U.S. paper size measuring 11 by 17 inches. The definition of A0 paper size is a $\sqrt{2}$ rectangle with area 1 square meter (accurate to within 1 square centimeter).

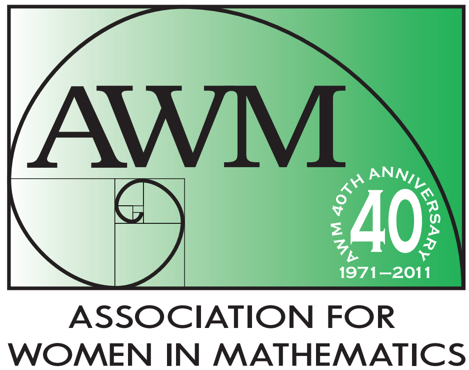

Figure 4. A golden ratio spiral incorporated into the logo of the Association for Women in Mathematics (www.awm-math.org); reprinted with permission from the AWM.

such a rectangle placed side by side with a square together form a larger rectangle with the same proportion. That is, a golden rectangle "added" to a square "equals" a golden rectangle. Repeatedly adding squares in this manner to form larger and larger golden rectangles, combined with drawing the appropriate quarter circle in each square, results in a well-known construction, often called a golden spiral,[4] which has been used in various images over the years, including the logo of the AWM (Figure 4). Again, we are barely scratching the surface of the mathematical and aesthetic properties of the golden ratio; entire books have been based on this topic, and indeed, we will consider a few such books below.

3. Vertex-to-Face Dual Pairings

Regarding the vertex-to-face pairings, let us first discuss the three out of the five ratios that are well documented. For the self-dual tetrahedron, the ratio is $1 : 3$ (we take the ratio from the inner polyedron's edge length to that of the outer polyhedron). When the octahedron is inside the cube, the ratio is $1 : \sqrt{2}$, which is, interestingly enough, the same as the edge-to-edge ratio for these two polyhedra (although in the case of the edge-to-edge pairing, it is the octahedron that has the larger edge length than the cube). When the icosahedron is inside the dodecahedron, the ratio is $\phi^2 : \sqrt{5}$. Here it may be a bit surprising that the inner icosahedron has a larger edge length than the outer dodecahedron, but this can be seen in Figure 2(b).

The two vertex-to-face ratios that seem to be missing from the literature occur when the cube is inside the octahedron, $\sqrt{2} : 3$, and when the dodecahedron is inside the icosahedron, $\phi : 3$. It is as if someone at some point in time failed to realize (or perhaps ignored the fact)

[4] Strictly speaking, this construction gives an approximation of a true golden spiral, the logarithmic spiral whose continuously increasing radius scales by a factor of ϕ for each 90° rotation. See [13], which provides more examples of spirals based on the golden ratio, as well as rigorous mathematical analyses thereof.

that there are two different configurations with the vertex-to-face duals (except in the case of the self-dual tetrahedron), and then this error of omission was repeated in the literature.[5]

Recall the elegant triplet of edge-length ratios described above for the edge-to-edge dual pairings, $1 : 1$, $1 : \sqrt{2}$, and $1 : \phi$. A related and similarly elegant sequence is given by the vertex-to-face edge-length ratios for the three cases where the outer dual polyhedra are composed of triangular faces, namely where the circumscribing polyhedra are the tetrahedron, the octahedron, and the icosahedron, respectively; this elegant sequence is $1 : 3$, $\sqrt{2} : 3$, and $\phi : 3$. Each of these three ratios has the same denominator, 3, and the numerators are the same important numbers as the denominators of the corresponding ratios in the edge-to-edge case.

4. Errors of Commission and Omission in the Literature

We next give a more specific overview of the appearance of these ratios in the literature. We have investigated both printed and online sources, using electronic searches, the "old school" method of visiting good math libraries and scanning through book after book on the relevant shelves, and the timeless method of contacting trusted colleagues who are more expert in appropriate fields. Again, the basic upshot is that all three of the edge-to-edge ratios are well documented but only three out of the five vertex-to-face ratios are; $\sqrt{2} : 3$ and $\phi : 3$ are absent, at least in the sources we have found, including several sources in which they certainly should appear.[6]

Actually, we did find one source [9], an online article co-written by a professor of philosophy and a professor of mathematics, having all five vertex-to-face ratios, but they give the exact form for only the tetrahedron's ratio, providing merely decimal approximations for the remaining four. That is, while their decimal expansions are correctly indicated to nine or ten significant digits, only the ratio for the tetrahedron, 1/3, is expressed in non-decimal form. Moreover, the authors explicitly expose the fact that they do not realize what numbers are indicated by the decimals:

> While the ratio for the tetrahedron's dual is quite elegant, the edge length ratios for the other regular polyhedra are not so elegant: .7071067810 of the edges of the octahedron inscribed in a cube; .4714045206 of the edges of the cube embedded in the octahedron; 1.17082094 of the edges of the icosahedron embedded in the dodecahedron; and .5393446629 of the edges of the dodecahedron inscribed in the icosahedron. [9]

While credit is due to Gier and Adele for having correctly provided close approximations of all five ratios, the whole point of the present article is that all these ratios are, in fact, "quite elegant"![7] What is required for one to see this, of course, is to identify the ratios in exact form in terms of the appropriate irrational numbers, such as the square root of two and the golden ratio. It should be pointed out that, depending on one's method, finding the exact form may require some subtleties in computation. For example, consider that when computing $\phi : 3$ with the various methods we have used, we are always faced with an expression whose simplest

[5] We implore the reader to contact us if she or he knows of the appearance of these ratios anywhere in the literature (except in the ways discussed below, which the reader will see are far from complete).

[6] We feel that it is important to admit that while we did investigate some non-English writings, we attempted an exhaustive search only for works in English.

[7] According to footnote 1a on the website [9], credit for the actual calculation is due to John Woll, a mathematics professor at Western Washington University.

	edge-length ratios	
solids	edge-to-edge	face-to-vertex
tetrahedron-tetrahedron	$1:1$	$1:1/2$ [sic]
cube-octahedron	$1:\sqrt{2}$	$1:1/\sqrt{2}$
dodecahedron-icosahedron	$1:\phi$	$\phi^2:\sqrt{5}$

TABLE 1. Edge-length ratios for Platonic solids and their duals, after [7].

form might best be expressed as $\frac{\sqrt{6+2\sqrt{5}}}{6}$ (see equation (5.4) below), which may not seem very elegant at first sight.

The only source, other than the aforementioned one using decimals, that at least implies knowledge of the $\sqrt{2}:3$ ratio is [2], a .pdf file available online comprising a classroom activity worksheet involving Platonic solids and their duals. After listing the dihedral angles of each solid, the worksheet then challenges the student to compute two vertex-to-face edge-length ratios: that between a tetrahedron and its dual, and that between a cube and its enclosing dual octahedron. (We, too, used dihedral angles to determine the vertex-to-face ratios, as we discuss in the next section.) The answers to these questions do not appear on the worksheet, but to be fair, we can only assume that its author(s) did calculate the exact forms of the answers ($1:3$ and $\sqrt{2}:3$). Note, however, that this worksheet makes no mention of the vertex-to-face edge-length ratio between a dodecahedron and its enclosing dual icosahedron; thus, there is no reason to infer that its author(s) calculated the $\phi:3$ ratio.

Other than these two websites, no other source that we have found (online or printed) indicates any knowledge whatsoever of the dual edge-length ratios $\sqrt{2}:3$ and $\phi:3$. Moreover, even if we have missed other appearances, it must be stressed that there are some in-print publications that really should include these two ratios, based on their subject matter. Stated differently, the fact that certain sources do not make mention of these two ratios is strong evidence that the authors were unaware of their existence. Let us be more specific with this contention.

The most thorough treatment we have found in print of the edge-length ratios for Platonic solids and their duals appears in [6] and [7], the former an article and the latter a book. Table 1 appears in each of them. The first issue to address is the mistake listed for the "tetrahedron-tetrahedron face-to-vertex" edge-length ratio, which should be $1:1/3$ instead of $1:1/2$ (the erroneous ratio appears in both [6] and [7]). In the context of the present article, however, the more important observation is that only three face-to-vertex ratios are shown, when in fact there should be two each for the "cube-octahedron" and "dodecahedron-icosahedron" dual relationships. The following text, which appears a few pages before this table in [7], may explain why only three face-to-vertex ratios are shown:

> The cube has six faces while the octahedron has six vertices. If a vertex is constructed at the center of each face of the cube and these vertices are connected together by edges, an octahedron is formed as shown in Figure 4.3 [analogous to our Figure 1(a)]. Similarly, the octahedron has eight faces and the cube has eight vertices. Constructing a vertex at the center of each face of the octahedron will produce a cube. Repeating this process produces smaller and smaller cubes and octahedra which are rescaled by a constant factor. [7, p. 26]

While it is not completely clear what the phrase "rescaled by a constant factor" indicates precisely, taking this whole statement together with the fact that only three face-to-vertex ratios appear in the table would suggest that Dunlap was unaware that there are two different ratios for each of the "cube-octahedron" and "dodecahedron-icosahedron" dual relationships, depending on which polyhedron is on the outside and which is on the inside. In any case, the two edge-length ratios that are at the crux of the present article, $\sqrt{2} : 3$ and $\phi : 3$, do not appear in [6] or [7].[8]

Next let us consider [12], a widely acclaimed book comprising an accessible account of the golden ratio and its appearance in a variety of contexts. Livio's text is a fascinating read, and he does a great job handling a broad range of truths and myths surrounding the golden ratio. He does include a discussion of dual Platonic solids, although only vertex-to-face dual pairings are described, probably due to a desire for accessibility, and he provides only one edge-length ratio, namely $\phi^2 : \sqrt{5}$. Livio's description of this ratio as it pertains to the icosahedron and dodecahedron is analogous to Dunlap's description of the cube and octahedron quoted above, in that Livio does not specify which polyhedron is inside the other:

> If we connect the centers of all the faces of the cube, we obtain an octahedron (Figure 21) [analogous to our Figure 1(a)], while if we connect the centers of the faces of an octahedron, we obtain a cube. The same procedure can be applied to map an icosahedron into a dodecahedron and vice versa, and the ratio of the edge lengths of the two solids (one embedded in the other) that are obtained can again be expressed in terms of the Golden Ratio, as $\phi^2/\sqrt{5}$. [12, p. 71]

It seems to us that $\phi : 3$ would have been a more elegant choice than $\phi^2 : \sqrt{5}$ to have provided at the end of this passage (assuming Livio's knowledge of the former, which is impossible to ascertain from [12]), although this is of course a matter of taste. More to the point, the fact is that neither $\phi : 3$ nor $\sqrt{2} : 3$ appear in Livio's discussion of edge-length ratios between dual Platonic solids in [12].

While Livio considers only vertex-to-face dual pairings, there are some other sources that consider only edge-to-edge dual pairings. One example is Kappraff [11], which contains both the cube/octahedron ratio, $1 : \sqrt{2}$, and the dodecahedron/icosahedron ratio, $1 : \phi$; the $1 : 1$ ratio for the self-dual tetrahedron is not explicitly cited, presumably because it is so obvious. As for other sources, the edge-to-edge dual ratio for the dodecahedron and icosahedron, $1 : \phi$, also appears on a wide variety of websites, which is not at all surprising, given that many people espouse the appearance of the golden ratio in various contexts. The MathWorld site on the dodecahedron [1] mentions the $1 : \phi$ ratio, but unfortunately there is an error on it.

The MathWorld site on the dodecahedron shows images of the usual three dual relationships between it and the icosahedron, i.e., images analogous to those in our Figure 2 above, but no attempt is made to distinguish between them in terms of edge-to-edge or vertex-to-face pairings. While there is nothing inherently wrong with this, the quote that directly follows their figures is problematic:

> The dual polyhedron of a dodecahedron with unit edge lengths is an icosahedron with edge lengths ϕ, where ϕ is the golden ratio. As a result, the centers

[8] We have sent Dr. Dunlap our description of his writings via email, in order to get his take on these issues, and he responded graciously, including: "I think that it is important to point out the additional vertex-to-face ratios that do not seem to have appeared anywhere." It is only fair for us to note that Dr. Dunlap had already caught the "1 : 1/2" error himself while working on a soon-to-be-published Turkish translation of his book.

> of the faces of an icosahedron form a dodecahedron, and vice versa (Steinhaus 1999, pp. 199–201). [1]

These two sentences taken together confuse the ratio based on the edge-to-edge dual pairing with the geometry of the vertex-to-face pairing. Furthermore, the cause-and-effect relationship between the first sentence and the second is quite misleading. "As a result" should not be used here, since there is no inherent relationship between the golden ratio fact and the ability of the center of the icosahedron's faces to form a dodecahedron (for that matter, Steinhaus [14, pp. 199–201] makes no mention whatsoever of the golden ratio in his discussion of dual polyhedra). That a website would be in error is no surprise, even one in the Wolfram domain, which is usually quite accurate (and useful). We mention it only because this is yet another example of incomplete and/or inaccurate information in the context of the edge-length ratios between dual Platonic solids.

There is a plethora of sources that discuss Platonic solids and their duals—too many to attempt an exhaustive list here—but most are silent on edge-length ratios, vertex-to-face or edge-to-edge. Some authors approach the topic from a certain point of view, e.g., model making [16] or paper folding [10], and so their focus is elsewhere. Many authors are motivated by the wonders of the golden ratio; besides [7] and [12] we should also explicitly mention [15], an English translation, recently published by the MAA in its Spectrum Series, of a German text. With this book, as with every other we have seen on the golden ratio besides [7] and [12], a detailed mathematical topic such as edge-length ratios between Platonic solids and their duals seems to be a bit beyond the scope. We looked at authors approaching Platonic solids from a design point of view, such as Critchlow [4], but such authors already have enough fascinating content generated by the art/math interface.[9] Articles in mathematics journals focus on very specific results, usually at a much higher level of mathematics, not surprisingly. And authors writing more traditional mathematical texts on polyhedra have so much other wonderful and rich mathematics to discuss in terms of polyhedra that edge-length ratios between duals get left out.

There are two notable examples in the last category. First let us consider a fairly recent text on polyhedra [5], which is an excellent resource in many respects. Interestingly enough, Cromwell does not discuss polar reciprocation. The edge-to-edge dual pairings are shown, but they are discussed from the standpoint of compounds of polyhedra, defined thusly:

> A *compound* polyhedron is a set of distinct polyhedra, called the *components* of the compound, which are placed together so that their centers coincide. [5, p. 359]

So the fact is that [5] does show figures analogous to our Figures 1(c), 2(c), and 3(b), but they are labeled "Compounds of the Platonic solids" [5, Figure 4.8 on p. 153]. Again, and more to the point of the present article, edge-length ratios between Platonic solids and their duals are not discussed at all.

Coxeter published several books on geometry, polyhedra and polytopes (higher-dimensional versions of polyhedra), such as [3]. While duals are indeed discussed, their edge-length ratios are not. For Coxeter and Cromwell, as for such historical giants as Kepler and Descartes, who also worked on the fundamentals of polyhedra, the primary focus was to find, categorize, and describe the wide array of polyhedra beyond the Platonic solids, so this might explain why this particular fact about edge-length ratios between duals has apparently gone unnoticed.

[9] We also investigated the considerable writings and drawings of R. Buckminster Fuller, best known for his work with geodesic domes, but Fuller was much more focused on engineering and societal concerns.

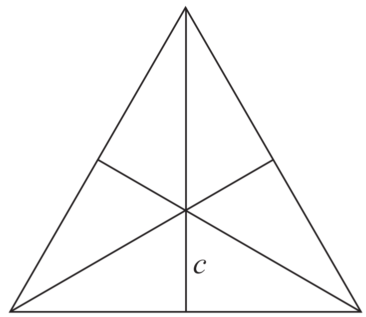

FIGURE 5. An equilateral triangle, with its three angle bisectors intersecting at its centroid; if the triangle has unit edge length, then $c = 1/(2\sqrt{3})$.

5. A Simple Mathematical Story, But Not Such an Obvious Ending

The last step(s) of the computation we show in this section might help to explain the gap in the literature described above. The most direct way we know to find the edge-length ratio between a regular dodecahedron and its dual enclosing icosahedron, shown above in Figure 2(a), is via the law of cosines, using the fact that the dihedral angle of the icosahedron is $\theta = \cos^{-1}\left(-\sqrt{5}/3\right)$.[10] Take the icosahedron to have unit edge length, and consider two of its adjacent triangular faces sharing a common edge. The distance, d, between the centroids of these two faces yields the desired result.

The line segment between the two centroids can be viewed as one side of an isosceles triangle, such that the angle opposite this line segment is the dihedral angle of the icosahedron. The other two sides, i.e., the equal sides of the isosceles triangle, each correspond to the line segment whose length is labeled c in Figure 5. Using this figure along with well-known facts about 30°-60°-90° triangles and similar triangles, it is easy to verify that $c = \frac{1}{2\sqrt{3}}$. Alternatively, one can argue that c must be 1/3 of the height of the equilateral triangle, by considering that the coordinates of the centroid are obtainable by averaging the coordinates of the three vertices.

The law of cosines now yields

$$d^2 = c^2 + c^2 - 2c^2\cos(\theta) \tag{5.1}$$

$$= 2c^2(1 + \sqrt{5}/3) \tag{5.2}$$

$$= \left(3 + \sqrt{5}\right)/18 \tag{5.3}$$

$$\Rightarrow d = \sqrt{\left(3 + \sqrt{5}\right)/18}\ . \tag{5.4}$$

What is the simplest form of this last expression? Rationalizing the denominator yields $d = \frac{\sqrt{6+2\sqrt{5}}}{6}$, which, although not terribly nasty, may not seem very elegant.

Coaxing $\phi/3$ to appear out of this last expression requires a working knowledge of the golden ratio, both its value and also an identity that is a direct result of the fact, mentioned above, that a golden rectangle, $1 : \phi$, appended to a square, $\phi : \phi$, produces another golden rectangle, $\phi : \phi + 1$. This leads to the useful identity $\phi^2 = \phi + 1$ (which can also be solved to obtain the value of the golden ratio, $\phi = (1 + \sqrt{5})/2$). Now we may proceed with the final step(s)

[10] Dihedral angles of Platonic solids are readily available; see, e.g., the CRC tables [17, p. 359].

required to find the elegant form of the key vertex-to-face ratio of the present article:

$$d = \sqrt{\frac{1}{18}\left(3+\sqrt{5}\right)} \tag{5.5}$$

$$= \frac{1}{3}\sqrt{\frac{1+\sqrt{5}}{2}+1} \tag{5.6}$$

$$= \frac{1}{3}\sqrt{\phi^2} \tag{5.7}$$

$$= \frac{\phi}{3}. \tag{5.8}$$

We can therefore conclude that the vertex-to-face edge-length ratio between a dodecahedron and its enclosing dual icosahedron is $\frac{\phi}{3} : 1$, i.e., $\phi : 3$. We also note that, while the above mathematics is quite basic, it may not occur to the majority of mathematicians to take these final steps. (One would first need to suspect the appearance of the golden ratio, and additionally one would need to have at least some experience working with it.) Indeed, this might help to explain why the $\phi : 3$ ratio seems to be absent from the literature, until now.

6. Conclusion

We have investigated the appearance in the literature of edge-length ratios between dual Platonic solids, considering both edge-to-edge dual pairings (resulting from polar reciprocation about the midsphere) and vertex-to-face dual pairings (resulting from polar reciprocation about either the circumscribing sphere or the inscribing sphere). Our focus is on the latter, of which there are five. Two of the five vertex-to-face edge-length ratios are virtually absent from the literature, at least as far as we have uncovered: $\phi : 3$ appears only in (approximate) decimal form [9]; $\sqrt{2} : 3$ appears in decimal form [9] and as the unstated answer of a question [2]. We have attempted an exhaustive search of English-language offerings, both in print and online, but of course we may have missed something. For example, there may be some specialized publication on crystallography that includes these ratios.

While hard to imagine, we may be the first to notice (or at least to put down in writing) the elegant triplet of ratios, $1 : 3$, $\sqrt{2} : 3$, and $\phi : 3$, given by the edge-length ratios for the three cases where the outer dual polyhedra are composed of triangular faces, i.e., where the circumscribing dual polyhedra are the regular tetrahedron, octahedron, and icosahedron, respectively. The first of these is certainly well-known, and it is the only rational ratio of the five vertex-to-face edge-length ratios. The second has surely been discovered before, since a question on [2] leads the student to try to find it. The third is the only one that seems to be truly absent from the literature (except in decimal form, as we have described). That such a fundamental ratio in geometry—and such a clear and clean appearance of the golden ratio in the Platonic solids—would be conspicuously absent from the literature seems unbelievable, even now as we complete this article after scouring sources in vain for the result.

References

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Department of Mathematical Sciences, Sweet Briar College, Sweet Briar, VA 24595
E-mail address: `wassell@sbc.edu`

PO Box 223, Sweet Briar, VA 24595
E-mail address: `benito11@sbc.edu`