PERIODIC REPRESENTATIONS FOR CUBIC IRRATIONALITIES

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ABSTRACT. In this paper we present some results related to the problem of finding periodic representations for algebraic numbers. In particular, we analyze the problem for cubic irrationalities. We show an interesting relationship between the convergents of bifurcating continued fractions related to a couple of cubic irrationalities, and a particular generalization of the Rédei polynomials. Moreover, we give a method to construct a periodic bifurcating continued fraction for any cubic root paired with another determined cubic root.

1. INTRODUCTION

In 1839 Hermite [13] posed to Jacobi the problem of finding methods for writing numbers that reflect special algebraic properties, i.e., finding periodic representations for algebraic numbers. Continued fractions completely solve this problem for every quadratic irrationality. This is the only known answer, indeed it has not yet been found a method in order to give a periodic representation for every algebraic irrationality of order greater than two. It seems natural to attempt the resolution of the Hermite problem researching some generalization of continued fractions. The first effort in this sense is due to Euler [7] in 1749, whose algorithm can provide periodic representations for cubic irrationalities. Successively, the algorithm was modified by Jacobi [15] in 1868 and extended to any algebraic irrationalities by Perron [20] in 1907 (for a complete survey about the Jacobi–Perron algorithm see [4]). During the years this generalization and some variations of the continued fractions have been deeply studied. For example Daus [6] developed the Jacobi algorithm for a particular couple of cubic irrationalities, connecting this study with the cubic Pell equation, and Lehmer [16] examined the convergence of particular periodic expansions. Further developments on the Jacobi–Perron algorithm can be found in [5, 8, 9, 11, 12, 14, 22].

In this paper, we focus only on the Jacobi algorithm and we study periodic representations and approximations for cubic irrationalities. In particular, introducing a generalization of the Rédei rational functions [21], we provide periodic representations for every couple $(\sqrt[3]{d^2}, \sqrt[3]{d})$ depending on a parameter z which can be any integer. Choosing different values for z, it is possible to obtain different periodic expansions and approximations for these irrationalities. Moreover, such representation has the advantage of having a small period making it easy to handle. Indeed, a problem of continued fractions and their generalization is the length of the period, which can be very large. Furthermore, in the case of the cubic irrationalities, periodicity is not guaranteed. As pointed out in [23], it seems that the Jacobi algorithm is not periodic for some couple of cubic irrationalities as, e.g., $(\sqrt[3]{3}, \sqrt[3]{9})$ (however it would be possible to find a cubic irrationality α such that the expansion of $(\sqrt[3]{3}, \alpha)$ is periodic). Thus, the possibility to obtain a periodic expansion for every couple $(\sqrt[3]{d^2}, \sqrt[3]{d})$ appears an important result in order to overcome the problems concerning the periodicity of the Jacobi algorithm.

The Rédei rational functions, generalized in the next section, arise from the development of $(z + \sqrt{d})^n$, where z is an integer and d is a nonsquare positive integer. One can write

$$(z+\sqrt{d})^n = N_n(d,z) + D_n(d,z)\sqrt{d}, \tag{1.1}$$

where

$$N_n(d,z) = \sum_{k=0}^{[n/2]} \binom{n}{2k} d^k z^{n-2k}, \quad D_n(d,z) = \sum_{k=0}^{[n/2]} \binom{n}{2k+1} d^k z^{n-2k-1}.$$

The Rédei rational functions $Q_n(d,z)$ are defined by

$$Q_n(d,z) = \frac{N_n(d,z)}{D_n(d,z)}, \quad \text{for all} \quad n \geq 1. \tag{1.2}$$

Their multiplicative property is well-known,

$$Q_{nm}(d,z) = Q_n(d, Q_m(d,z)),$$

for any couple of indexes n, m. Thus Rédei functions are closed with respect to composition and satisfy the commutative property

$$Q_n(d, Q_m(d,z)) = Q_m(d, Q_n(d,z)).$$

The Rédei rational functions reveal their utility in several fields of number theory. Given a finite field $\mathbb{F}_q$, of order q, and $\sqrt{d} \notin \mathbb{F}_q$, then $Q_n(d,z)$ is a permutation of $\mathbb{F}_q$ if and only if $(n, q+1) = 1$ [17, p. 44]. Another recent application of these functions provides a way to find a new bound for multiplicative character sums of nonlinear recurring sequences [10]. Moreover, they can be used in order to generate pseudorandom sequences [24] and to create a public key cryptographic system [18]. In a previous work [3] we have seen how Rédei rational functions can be used in order to generate solutions of the Pell equation in an original way, applying them in a totally new field with respect to the classic ones. Furthermore, in [1] we have introduced these functions as convergents of certain periodic continued fractions which always represent square roots. Here we generalize this construction for cubic roots, providing a generalization of the Rédei rational functions in order to obtain periodic representations for every cubic root.

2. The Jacobi Algorithm and Bifurcating Continued Fractions

In this section we briefly recall the Jacobi algorithm. It is a generalization of the Euclidean algorithm used for constructing the classic continued fractions. In this generalization instead of representing a real number by an integer sequence, a couple of real numbers are represented by a couple of integer sequences. The algorithm that provides such integer sequences is

$$\begin{cases} a_n = [x_n] \\ b_n = [y_n] \\ x_{n+1} = \dfrac{1}{y_n - [y_n]} \\ y_{n+1} = \dfrac{x_n - [x_n]}{y_n - [y_n]}, \end{cases} \tag{2.1}$$

$n = 0, 1, 2, \ldots$, for any couple of real numbers $x = x_0$ and $y = y_0$. We can retrieve x and y from the sequences $(a_n)_{n=0}^{+\infty}$ and $(b_n)_{n=0}^{+\infty}$ using

$$\begin{cases} x_n = a_n + \dfrac{y_{n+1}}{x_{n+1}} \\ y_n = b_n + \dfrac{1}{x_{n+1}} \end{cases} \tag{2.2}$$

$n = 0, 1, 2, \ldots$, indeed by equations (2.1) it follows

$$a_n + \frac{y_{n+1}}{x_{n+1}} = a_n + \cfrac{\cfrac{x_n - a_n}{y_n - b_n}}{\cfrac{1}{y_n - b_n}} = a_n + x_n - a_n = x_n, \quad \text{for all} \quad n \geq 0$$

$$b_n + \frac{1}{x_{n+1}} = b_n + \cfrac{1}{\cfrac{1}{y_n - b_n}} = b_n + y_n - b_n = y_n, \quad \text{for all} \quad n \geq 0.$$

Therefore, the real numbers x and y are represented by the sequences as follows:

$$x = a_0 + \cfrac{b_1 + \cfrac{1}{a_2 + \cfrac{\left(b_3 + \cfrac{1}{\ddots}\right)}{\left(a_3 + \cfrac{\ddots}{\ddots}\right)}}}{a_1 + \cfrac{b_2 + \cfrac{1}{\left(a_3 + \cfrac{\ddots}{\ddots}\right)}}{a_2 + \cfrac{\left(b_3 + \cfrac{1}{\ddots}\right)}{\left(a_3 + \cfrac{\ddots}{\ddots}\right)}}} \quad \text{and} \quad y = b_0 + \cfrac{1}{a_1 + \cfrac{b_2 + \cfrac{1}{\left(a_3 + \cfrac{\ddots}{\ddots}\right)}}{a_2 + \cfrac{\left(b_3 + \cfrac{1}{\ddots}\right)}{\left(a_3 + \cfrac{\ddots}{\ddots}\right)}}} \tag{2.3}$$

Two objects representing the numbers x and y are called *ternary* or *bifurcating* continued fractions. We call *partial quotients* the integers a_i and b_i, for $i = 0, 1, 2, \ldots$. We can briefly write the bifurcating continued fraction (2.3) with the notation $[\{a_0, a_1, a_2, \ldots\}, \{b_0, b_1, b_2, \ldots\}]$ and we can introduce the notion of convergent-like for the classic continued fraction. (For a complete survey of the Jacobi–Perron algorithm see [4]). The finite bifurcating continued fraction

$$[\{a_0, a_1, \ldots, a_n\}, \{b_0, b_1, \ldots, b_n\}] = \left(\frac{A_n}{C_n}, \frac{B_n}{C_n}\right), \quad \text{for all} \quad n \geq 0$$

is called nth *convergent*, where the integers A_n, B_n, C_n are defined by the following recurrent relations (see, e.g., [11, 12]) for every $n \geq 3$:

$$\begin{cases} A_n = a_n A_{n-1} + b_n A_{n-2} + A_{n-3} \\ B_n = a_n B_{n-1} + b_n B_{n-2} + B_{n-3} \\ C_n = a_n C_{n-1} + b_n C_{n-2} + C_{n-3}. \end{cases} \tag{2.4}$$

We can introduce the convergents of a bifurcating continued fraction using a matricial description, like for the classic continued fractions. It is easy to prove by induction that

$$\begin{pmatrix} a_0 & 1 & 0 \\ b_0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 & 0 \\ b_n & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} A_n & A_{n-1} & A_{n-2} \\ B_n & B_{n-1} & B_{n-2} \\ C_n & C_{n-1} & C_{n-2} \end{pmatrix} \tag{2.5}$$

$$\Updownarrow$$

$$\left(\frac{A_n}{C_n}, \frac{B_n}{C_n}\right) = [\{a_0, \ldots, a_n\}, \{b_0, \ldots, b_n\}].$$

Remark 2.1. *We observe that the algorithm (2.1) does not provide every bifurcating continued fraction expansion (2.3). Indeed, it is easy to prove that using equations (2.1) we always obtain $a_i \geq b_i$ for all $i \geq 1$. However, a bifurcating continued fraction can represent a couple of real numbers (i.e., the limit of the convergents exists and it is finite), although it is not obtained starting from the Jacobi algorithm.*

Every periodic bifurcating continued fraction converges to a couple of cubic irrationalities, but the viceversa is unproved. We do not know if, given any cubic irrationality, another cubic irrationality ever exists such that their bifurcating continued fraction expansion is periodic. Therefore, the Hermite problem is still open for any algebraic irrationalities, except for the quadratic case.

The properties of the Jacobi algorithm can be studied by using the characteristic polynomial of the matrices (2.5). For instance, if we consider the purely periodic bifurcating continued fraction $(\alpha, \beta) = [\{\overline{a_0, \ldots, a_n}\}, \{\overline{b_0, \ldots, b_n}\}]$, such fraction converges to cubic irrationalities related to the roots of the polynomial

$$\det \begin{pmatrix} A_n - x & A_{n-1} & A_{n-2} \\ B_n & B_{n-1} - x & B_{n-2} \\ C_n & C_{n-1} & C_{n-2} - x \end{pmatrix} = 0.$$

Moreover, it is possible to directly study the convergence, considering that in this case we can write $(\alpha, \beta) = [\{a_0, \ldots, a_n, \alpha\}, \{b_0, \ldots, b_n, \beta\}]$ and

$$\alpha = \frac{\alpha A_n + \beta A_{n-1} + A_{n-2}}{\alpha C_n + \beta C_{n-1} + C_{n-2}}, \quad \beta = \frac{\alpha B_n + \beta B_{n-1} + B_{n-2}}{\alpha C_n + \beta C_{n-1} + C_{n-2}}.$$

Similar considerations can be performed in the case of eventually periodic fractions.

The difficulties to prove an analog of the Lagrange Theorem (every quadratic irrational has periodic continued fraction expansion) arise by the fact that there are no explicit forms for cubic irrationalities. For this reason many different studies of the Jacobi algorithm has been performed. For example, the discussion of bifurcating continued fractions is related to the problem of finding units in cubic fields. Other ways involve the research of bound for the partial quotients and the convergents.

The Jacobi algorithm can also be approached studying a transformation of $\mathbb{R}^2$ into itself (see [22]), defined as follows:

$$T(\alpha,\beta)=\left(\frac{\beta}{\alpha}-\left[\frac{\beta}{\alpha}\right],\frac{1}{\alpha}-\left[\frac{1}{\alpha}\right]\right).$$

Using the following auxiliary maps over $\mathbb{R}^2$, defined by

$$\tau(\alpha,\beta)=\left[\frac{1}{\alpha}\right],\quad \eta(\alpha,\beta)=\left[\frac{\beta}{\alpha}\right],$$

the partial quotients of the Jacobi algorithm are determined by

$$\begin{cases} a_i=\tau(\alpha_i,\beta_i)\\ b_i=\eta(\alpha_i,\beta_i)\end{cases} \tag{2.6}$$

where $(\alpha_i,\beta_i)=T(\alpha_{i-1},\beta_{i-1})$, for $i=0,1,2,\dots$. Initializing the procedure with $\alpha_0=\dfrac{1}{x}$ and $\beta_0=\dfrac{y}{x}$, for any couple of real numbers x and y, the sequence of partial quotients provided by equation (2.6) coincides with the sequence provided by the Jacobi algorithm as presented in (2.1). In this way periodicity, convergence and other properties of the Jacobi algorithm can be found studying the transformation T which satisfies ergodic properties.

In the next section, we propose a different approach, studying the convergence properties of some polynomials and exploiting the fact that they satisfy linear recurrent relations.

3. Generalized Rédei Rational Functions and Periodic Representations of Cubic Roots

As we have seen in the introduction, Rédei rational functions are strictly connected to square roots. In this paragraph we propose a generalization of the Rédei polynomials related to every algebraic irrationality. Successively we will focus on cubic irrationalities, highlighting a connection with bifurcating continued fractions and the Hermite problem.

Instead of considering the expansion of $(z+\sqrt{d})^n$, we start analyzing the expansion of $(z+\sqrt[e]{d^{e-1}})^n$, where d is not an eth power. We observe that in the expansion of $(z+\sqrt[e]{d^{e-1}})^n$ we have coefficients for $\sqrt[e]{d^2},\sqrt[e]{d^3},\dots,\sqrt[e]{d^{e-1}}$. Thus, we use e polynomials in order to write its expansion:

$$(z+\sqrt[e]{d^{e-1}})^n=\mu_n(e,0,d,z)+\mu_n(e,1,d,z)\sqrt[e]{d}+\cdots+\mu_n(e,e-1,d,z)\sqrt[e]{d^{e-1}}, \tag{3.1}$$

where

$$\mu_n(k)=\mu_n(e,k,d,z)=\sum_{h=0}^{n}\binom{n}{eh-k}d^{(e-1)h-k}z^{n-eh+k}. \tag{3.2}$$

Using the $e\times e$ matrix

$$\begin{pmatrix} z & d & 0 & \dots & 0\\ 0 & z & d & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & z & d\\ 1 & 0 & 0 & \dots & z\end{pmatrix}, \tag{3.3}$$

whose characteristic polynomial $(x-z)^e - d^{e-1}$ has root of larger modulus $z+d^{(e-1)/e}$, defining $\mu_n(k) = \mu_n(e,k,d,z)$ for $k = 0,1,\dots,e-1$, we have

$$\begin{pmatrix} z & d & 0 & \dots & 0 \\ 0 & z & d & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & z & d \\ 1 & 0 & 0 & \dots & z \end{pmatrix}^n = \begin{pmatrix} \mu_n(0) & d\mu_n(e-1) & \dots & d\mu_n(1) \\ \mu_n(1) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & d\mu_n(e-1) \\ \mu_n(e-1) & \dots & \mu_n(1) & \mu_n(0) \end{pmatrix}.$$

Example 3.1. *When $e = 3$, we have*

$$\begin{pmatrix} z & d & 0 \\ 0 & z & d \\ 1 & 0 & z \end{pmatrix}^n = \begin{pmatrix} \mu_n(0) & d\mu_n(2) & d\mu_n(1) \\ \mu_n(1) & \mu_n(0) & d\mu_n(2) \\ \mu_n(2) & \mu_n(1) & \mu_n(0) \end{pmatrix}.$$

Remark 3.2. *The sequences of polynomials μ_n are linear recurrent sequences. Indeed, they correspond to the entries of a matrix power, and so they recur with the characteristic polynomial of the resulting power matrix. In particular the sequence $(\mu_n(e,k,d,z))_{n=0}^{\infty}$ recurs with polynomial $(x-z)^e - d^{e-1}$.*

We can observe the convergence of the polynomials μ_n:

$$\lim_{n\to\infty} \frac{\mu_n(e,k,d,z)}{\mu_n(e,e-1,d,z)} = \sqrt[e]{d^{e-k-1}}, \quad k = 0,\dots,e-2.$$

Now, we focus our attention on the cubic case. In the next theorem we prove the convergence of the polynomials μ_n (since these polynomials will be used in the next paragraph) when $e = 3$.

Theorem 3.3. *Let d be a non-cube integer. Then*

$$\lim_{n\to\infty} \frac{\mu_n(3,0,d,z)}{\mu_n(3,2,d,z)} = \sqrt[3]{d^2}$$

$$\lim_{n\to\infty} \frac{\mu_n(3,1,d,z)}{\mu_n(3,2,d,z)} = \sqrt[3]{d}.$$

Proof. The sequence $(\mu_n(3,k,d,z))_{n=0}^{\infty}$ recurs with polynomial $(x-z)^3 - d^2$ having real root $\alpha_1 = z + \sqrt[3]{d^2}$ of larger modulus than the remaining roots α_2, α_3. By the Binet formula

$$\begin{cases} \mu_n(3,0,d,z) = a_1\alpha_1^n + a_2\alpha_2^n + a_3\alpha_3^n \\ \mu_n(3,1,d,z) = b_1\alpha_1^n + b_2\alpha_2^n + b_3\alpha_3^n \\ \mu_n(3,2,d,z) = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n. \end{cases}$$

Solving the systems

$$\begin{cases} a_1 + a_2 + a_3 = 1 \\ a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = z \\ a_1\alpha_1^2 + a_2\alpha_2^2 + a_3\alpha_3^2 = z^2 \end{cases} \quad \begin{cases} b_1 + b_2 + b_3 = 0 \\ b_1\alpha_1 + b_2\alpha_2 + b_3\alpha_3 = 0 \\ b_1\alpha_1^2 + b_2\alpha_2^2 + b_3\alpha_3^2 = d \end{cases} \quad \begin{cases} c_1 + c_2 + c_3 = 0 \\ c1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 = 1 \\ c_1\alpha_1^2 + c_2\alpha_2^2 + c_3\alpha_3^2 = 2z \end{cases}$$

we easily obtain

$$a_1 = \frac{1}{3}, \quad b_1 = \frac{1}{3\sqrt[3]{d}}, \quad c_1 = \frac{1}{3\sqrt[3]{d^2}}.$$

Finally,

$$\frac{\mu_n(3,0,d,z)}{\mu_n(3,2,d,z)} = \frac{a_1\alpha_1^n + a_2\alpha_2^n + a_3\alpha_3^n}{c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n} \to \frac{a_1}{c_1} = \sqrt[3]{d^2}$$
$$\frac{\mu_n(3,1,d,z)}{\mu_n(3,2,d,z)} = \frac{b_1\alpha_1^n + b_2\alpha_2^n + b_3\alpha_3^n}{c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n} \to \frac{b_1}{c_1} = \sqrt[3]{d}.$$

□

Now, we use the polynomials μ_n together with bifurcating continued fractions in order to approximate cubic roots. In particular we provide a periodic bifurcating continued fraction expansion for any cubic root, whose convergents are the ratios of these polynomials.

First of all, we study bifurcating continued fractions with rational partial quotients.

Remark 3.4. *In* [1] *and* [2], *we have studied algebraic properties of continued fractions with rational partial quotients. In this way, it is possible to obtain periodic expansions for quadratic irrationalities more handily. Furthermore, these continued fractions have interesting properties of approximations related to Rédei rational functions. At the same time, we proved that Newton and Padé provide approximations of square roots. Thus, it seems natural to use rational partial quotients with bifurcating continued fractions.*

If we consider a bifurcating continued fraction

$$\left[\left\{\frac{a_0}{b_0}, \frac{a_1}{b_1}, \dots\right\}, \left\{\frac{c_0}{d_0}, \frac{c_1}{d_1}, \dots\right\}\right],$$

the sequences A_n, B_n, C_n in (2.4) are rational numbers. Therefore, we can study the recurrence of numerators and denominators of such rational sequences. In the following theorem we provide the result only for the sequence A_n. Similar results clearly hold for B_n, C_n.

Lemma 3.5. *Given*

$$\left[\left\{\frac{a_0}{b_0}, \frac{a_1}{b_1}, \dots\right\}, \left\{\frac{c_0}{d_0}, \frac{c_1}{d_1}, \dots\right\}\right],$$

let $(A_n)_{n=0}^{\infty}, (B_n)_{n=0}^{\infty}, (C_n)_{n=0}^{\infty}$ *be the sequences such that*

$$\left[\left\{\frac{a_0}{b_0}, \dots, \frac{a_n}{b_n}\right\}, \left\{\frac{c_0}{d_0}, \dots, \frac{c_n}{d_n}\right\}\right] = \left(\frac{A_n}{C_n}, \frac{B_n}{C_n}\right)$$

for all $n \geq 0$. *Then* $A_n = \dfrac{s_n}{t_n}$ *for every* $n \geq 0$, *where*

$$\begin{cases} s_{-1} = 1/d_0, \quad s_0 = a_0, \quad s_1 = a_0a_1d_1 + b_0b_1c_1 \\ s_n = a_nd_ns_{n-1} + b_nb_{n-1}c_nd_{n-1}s_{n-2} + b_nb_{n-1}b_{n-2}d_nd_{n-1}d_{n-2}s_{n-3}, \quad n \geq 2 \end{cases}$$

and

$$\begin{cases} t_0 = b_0 \\ t_n = b_0\prod_{i=1}^n b_id_i, \quad n \geq 1. \end{cases}$$

Proof. We prove the theorem by induction. The verification of the inductive basis is straightforward.

Let us suppose the thesis is true for all the integers less or equal to $n-1$ and we prove it for n. Considering the recurrences (2.4) and the inductive hypothesis, we have

$$A_n = \frac{a_n}{b_n}A_{n-1} + \frac{c_n}{d_n}A_{n-2} + A_{n-3} = \frac{a_n}{b_n}\frac{s_{n-1}}{t_{n-1}} + \frac{c_n}{d_n}\frac{s_{n-2}}{t_{n-2}} + \frac{s_{n-3}}{t_{n-3}}$$

$$= \frac{a_n s_{n-1}}{b_n b_0 b_1 d_1 \cdots b_{n-1} d_{n-1}} + \frac{c_n s_{n-2}}{d_n b_0 b_1 d_1 \cdots b_{n-2} d_{n-2}} + \frac{s_{n-3}}{b_0 b_1 d_1 \cdots b_{n-3} d_{n-3}}$$
$$= \frac{a_n d_n s_{n-1} + b_n b_{n-1} c_n d_{n-1} s_{n-2} + b_n b_{n-1} b_{n-2} d_n d_{n-1} d_{n-2} s_{n-3}}{b_0 b_1 d_1 \cdots b_n d_n} = \frac{s_n}{t_n}.$$

□

Remark 3.6. *Even if we have posed the condition $s_{-1} = 1/d_0$, the sequence $(s_n)_{n=0}^{\infty}$ is an integer sequence, since*

$$s_2 = a_2 d_2 s_1 + b_2 b_1 c_2 d_1 s_0 + b_2 b_1 b_0 d_2 d_1.$$

It is possible to prove similar results for (B_n) and (C_n):

$$B_n = \frac{s'_n}{t'_n} \quad n \geq 0,$$

where $(s'_n)_{n=0}^{\infty}$ recurs as the sequence (s_n), but with initial conditions

$$\begin{cases} s'_0 = c_0 \\ s'_1 = a_1 c_0 + b_1 d_0 \\ s'_2 = b_1 b_2 c_0 c_2 + a_1 a_2 c_0 d_2 + a_2 b_1 d_0 d_2 \end{cases}$$

and

$$\begin{cases} t'_0 = d_0, \quad t'_1 = d_0 b_1 \\ t'_n = d_0 b_1 \prod_{i=2}^{n} b_i d_i, \quad n \geq 2. \end{cases}$$

Similarly we obtain

$$C_n = \frac{s''_n}{t''_n}, \quad n \geq 0,$$

where $(s''_n)_{n=0}^{\infty}$ recurs as (s_n) with initial conditions

$$\begin{cases} s''_0 = 1, \quad s''_1 = a_1 \\ s''_2 = b_1 b_2 c_2 + a_1 a_2 d_2 \end{cases}$$

and

$$\begin{cases} t''_0 = 1, \quad t''_1 = b_1 \\ t''_n = b_1 \prod_{i=2}^{n} b_i d_i, \quad n \geq 2. \end{cases}$$

We can conclude that we have the following expressions for the convergents of a bifurcating continued fraction with rational partial quotients:

$$\frac{A_0}{C_0} = \frac{s_0}{b_0 s''_0}, \quad \frac{A_n}{C_n} = \frac{s_n}{b_0 d_1 s''_n} \quad \text{for all} \quad n \geq 1$$

and

$$\frac{B_n}{C_n} = \frac{s'_n}{d_0 s''_n} \quad \text{for all} \quad n \geq 0,$$

where the sequences $(s_n), (s'_n), (s''_n)$ are integer sequences.

Now, we study the Hermite problem for cubic irrationalities, observing a connection between the polynomials μ_n and the bifurcating continued fractions. By using rational partial quotients we can give a periodic expansion for every couple of cubic irrationalities of the kind $(\sqrt[3]{d^2}, \sqrt[3]{d})$, whose approximations are provided by the polynomials μ_n. In order to do this, we recall that polynomials $\mu_n = \mu_n(e,k,d,z)$, where now we consider $e = 3$, have the following matricial representation:

$$\begin{pmatrix} z & d & 0 \\ 0 & z & d \\ 1 & 0 & z \end{pmatrix}^n = \begin{pmatrix} \mu_n(0) & d\mu_n(2) & d\mu_n(1) \\ \mu_n(1) & \mu_n(0) & d\mu_n(2) \\ \mu_n(2) & \mu_n(1) & \mu_n(0) \end{pmatrix}, \tag{3.4}$$

where we write the only dependence from k.

Theorem 3.7. *The periodic bifurcating continued fraction*

$$\left[\left\{z, \frac{2z}{d}, \overline{\frac{3dz}{z^3+d^2}, 3z, \frac{3z}{d}}\right\}, \left\{0, -\frac{z^2}{d}, \overline{-\frac{3z^2}{z^3+d^2}, -\frac{3dz^2}{z^3+d^2}, -\frac{3z^2}{d}}\right\}\right] \tag{3.5}$$

converges for every integer $z \neq 0$ to the couple of irrationals $(\sqrt[3]{d^2}, \sqrt[3]{d})$ and its convergents are the couple of rationals

$$\left(\frac{\mu_n(3,0,d,z)}{\mu_n(3,2,d,z)}, \frac{\mu_n(3,1,d,z)}{\mu_n(3,2,d,z)}\right),$$

for $\mu_n(e,k,d,z)$ polynomials defined in (3.2), $n \geq 1$.

Proof. By Theorem 3.3, we have only to prove that the convergents of (3.5) are

$$\left(\frac{\mu_n(3,0,d,z)}{\mu_n(3,2,d,z)}, \frac{\mu_n(3,1,d,z)}{\mu_n(3,2,d,z)}\right).$$

In this way the periodic bifurcating continued fraction (3.5) clearly converges to $(\sqrt[3]{d^2}, \sqrt[3]{d})$. For the sake of simplicity we specify only the dependence on k for the polynomials μ_n:

$$\mu_n(3,k,d,z) = \mu_n(k).$$

We will use the representation of the convergents showed in Lemma 3.5. We start by observing that in this case we have

$$b_0 = 1, \quad d_0 = 1, \quad d_1 = d$$

and

$$s_0 = a_0 = z = \mu_1(0), \quad s_1 = a_0a_1d_1 + b_0b_1c_1 = 2dz^2 - dz^2 = dz^2 = d\mu_2(0),$$
$$s_0' = c_0 = 0 = \mu_1(1), \quad s_1' = a_1c_0 + b_1d_0 = d = \mu_2(1),$$
$$s_0'' = 1 = \mu_1(2), \quad s_1'' = a_1 = 2z = \mu_2(2).$$

Therefore, for the convergents of (3.5) we initially have

$$\frac{s_0}{b_0s_0''} = \frac{\mu_1(0)}{\mu_1(2)}, \quad \frac{s_0'}{d_0s_0''} = \frac{\mu_1(1)}{\mu_1(2)},$$

$$\frac{s_1}{b_0d_1s_1''} = \frac{s_1}{ds_1''} = \frac{\mu_2(0)}{\mu_2(2)}, \quad \frac{s_1'}{d_0s_1''} = \frac{\mu_2(1)}{\mu_2(2)}.$$

Now, we prove by induction the following relation:

$$s_n = d^{\left[\frac{2(n+1)}{3}\right]}(d^2+z^3)^{\left[\frac{2n}{3}\right]}\mu_{n+1}(0), \quad \text{for all} \quad n \geq 2. \tag{3.6}$$

The inductive basis is straightforward, finding that

$$s_2 = d^2(d^2+z^3)\mu_3(0).$$

We now proceed with the induction, supposing true the thesis for every integer less or equal to $n-1$ and proving the thesis for n. Since the period of the fraction is 3, we have to discuss 3 cases for n, i.e.,

$$n \equiv 0 \pmod 3, \quad n \equiv 1 \pmod 3, \quad n \equiv 2 \pmod 3.$$

In this proof we only consider the case $n \equiv 0 \pmod 3$; for the other case the proof is similar. By Lemma 3.5 we know that

$$s_n = a_n d_n s_{n-1} + b_n b_{n-1} c_n d_{n-1} s_{n-2} + b_n b_{n-1} b_{n-2} d_n d_{n-1} d_{n-2} s_{n-3}.$$

Since $n \equiv 0 \pmod 3$, we have

$$\begin{cases} a_n = 3z, \quad b_n = 1, \quad c_n = -3dz^2, \quad d_n = z^3 + d^2, \\ a_{n-1} = 3dz, \quad b_{n-1} = z^3 + d^2, \quad c_{n-1} = -3z^2, \quad d_{n-1} = z^3 + d^2, \\ a_{n-2} = 3z, \quad b_{n-2} = d, \quad c_{n-2} = -3z^2, \quad d_{n-2} = d. \end{cases}$$

Thus, considering the inductive hypothesis, we have

$$\begin{aligned} s_n &= 3z(d^2+z^3)d^{\left[\frac{2n}{3}\right]}(d^2+z^3)^{\left[\frac{2n-2}{3}\right]}\mu_n(0) \\ &\quad - 3z^2 d(d^2+z^3)^2 d^{\left[\frac{2(n-1)}{3}\right]}(d^2+z^3)^{\left[\frac{2n-4}{3}\right]}\mu_{n-1}(0) \\ &\quad + (d^2+z^3)^3 d^2 d^{\left[\frac{2(n-2)}{3}\right]}(d^2+z^3)^{\left[\frac{2n-6}{3}\right]}\mu_{n-2}(0), \end{aligned}$$

i.e.,

$$\begin{aligned} s_n &= 3zd^{\left[\frac{2n}{3}\right]}(d^2+z^3)^{\left[\frac{2n+1}{3}\right]}\mu_n(0) \\ &\quad - 3z^2 d^{\left[\frac{2n+1}{3}\right]}(d^2+z^3)^{\left[\frac{2(n+1)}{3}\right]}\mu_{n-1}(0) \\ &\quad + (d^2+z^3)d^{\left[\frac{2(n+1)}{3}\right]}(d^2+z^3)^{\left[\frac{2n}{3}\right]}\mu_{n-2}(0). \end{aligned}$$

Since $n \equiv 0 \pmod 3$, i.e., $n = 3k$, we have the following identities

$$\left[\frac{2n}{3}\right] = \left[\frac{2(n+1)}{3}\right] = \left[\frac{2n+1}{3}\right].$$

Indeed,

$$\left[\frac{2 \cdot 3k}{3}\right] = 2k, \quad \left[\frac{2(3k+1)}{3}\right] = \left[2k + \frac{2}{3}\right] = 2k, \quad \left[\frac{2 \cdot 3k + 1}{3}\right] = \left[2k + \frac{1}{3}\right] = 2k.$$

Therefore, we have

$$s_n = d^{\left[\frac{2(n+1)}{3}\right]}(d^2+z^3)^{\left[\frac{2n}{3}\right]}(3z\mu_n(0) - 3z^2\mu_{n-1}(0) + (d^2+z^3)\mu_{n-2}(0))$$

and remembering the recurrence relation involving the polynomials μ_n, the equation (3.6) follows. It is possible to prove in a similar way the formulas

$$s'_n = d^{\left[\frac{2n-1}{3}\right]}(d^2+z^3)^{\left[\frac{2n}{3}\right]}\mu_{n+1}(1), \quad \text{for all} \quad n \geq 2,$$

$$ds''_n = d^{\left[\frac{2(n+1)}{3}\right]}(d^2+z^3)^{\left[\frac{2n}{3}\right]}\mu_{n+1}(2), \quad \text{for all} \quad n \geq 2.$$

Hence, for the couple of convergents of the fraction (3.5) we obtain

$$\frac{s_n}{b_0 d_1 s''_n} = \frac{s_n}{ds''_n} = \frac{\mu_{n+1}(0)}{\mu_{n+1}(2)}, \quad \text{for all} \quad n \geq 2,$$

$$\frac{s'_n}{d_0 s''_n} = \frac{s'_n}{s''_n} = \frac{\mu_{n+1}(1)}{\mu_{n+1}(2)}, \quad \text{for all} \quad n \geq 2$$

and, considering what has been observed for indexes $n = 0, 1$, the proof is complete. □

In the previous theorem we saw an important result about cubic roots. Indeed, we found a periodic representation in the sense of the Hermite problem providing rational approximations for cubic roots related to a generalization of the Rédei rational functions. It is interesting to note that the previous representation is valid for every choice of z integer, providing in this way different rational approximations for the same cubic root. Moreover, the period of (3.5) is really short and it is in general shorter than the period of the bifurcating continued fractions obtained from the Jacobi algorithm. We highlight such considerations and the differences between our representation and the Jacobi one in the next examples.

Example 3.8. *Let us consider the cubic roots* $(\sqrt[3]{16}, \sqrt[3]{4})$. *Choosing, for example,* $z = [\sqrt[3]{16}] = 2$, *by the previous theorem, we have*

$$(\sqrt[3]{16}, \sqrt[3]{4}) = \left[\left\{2, 1, \overline{1, 6, \frac{3}{2}}\right\}, \left\{0, -1, \overline{-\frac{1}{2}, -2, -3}\right\}\right],$$

against the Jacobi algorithm which seems to have non–periodicity. Indeed, evaluating the first 1000 partial quotients of the expansion of $(\sqrt[3]{16}, \sqrt[3]{4})$ *no periodic patterns appear with the Jacobi algorithm.*

Example 3.9. *Let us consider the cubic roots* $(\sqrt[3]{25}, \sqrt[3]{5})$. *In this case the Jacobi algorithm provides a periodic expansion of period 6 and pre-period 7, given by*

$$[\{2, 1, 3, 2, 1, 1, 7, \overline{1, 1, 2, 3, 1, 6}\}, \{1, 1, 1, 0, 0, 0, 0, \overline{0, 0, 0, 0, 1, 2}\}]$$

against our representation of period 3 and pre-period 2, which, choosing for example, $z = [\sqrt[3]{5}] = 1$, *is*

$$\left[\left\{1, \frac{2}{5}, \overline{\frac{15}{26}, 3, \frac{3}{5}}\right\}, \left\{0, -\frac{1}{5}, \overline{-\frac{3}{26}, -\frac{15}{26}, -\frac{3}{5}}\right\}\right].$$

Finally, previous representation allow us to retrieve periodic representations and approximations by using linear recurrent sequences for a vast class of cubic irrationalities. The sequences $(A_n)_{n=0}^{+\infty}, (B_n)_{n=0}^{+\infty}, (C_n)_{n=0}^{+\infty}$ determine the convergents of the fraction (3.5), if we consider the matrix

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} A_n & A_{n-1} & A_{n-2} \\ B_n & B_{n-1} & B_{n-2} \\ C_n & C_{n-1} & C_{n-2} \end{pmatrix} = \begin{pmatrix} \tilde{A}_n & \tilde{A}_{n-1} & \tilde{A}_{n-2} \\ \tilde{B}_n & \tilde{B}_{n-1} & \tilde{B}_{n-2} \\ \tilde{C}_n & \tilde{C}_{n-1} & \tilde{C}_{n-2} \end{pmatrix}.$$

it is easy to study the convergence of $\frac{\tilde{A}_n}{\tilde{C}_n}$ and $\frac{\tilde{B}_n}{\tilde{C}_n}$. In fact, we have

$$\lim_{n\to\infty} \frac{\tilde{A}_n}{\tilde{C}_n} = \lim_{n\to\infty} \frac{a_{00}A_n + a_{01}B_n + a_{02}C_n}{a_{20}A_n + a_{21}B_n + a_{22}C_n} = \lim_{n\to\infty} \frac{a_{00}\frac{A_n}{C_n} + a_{01}\frac{B_n}{C_n} + a_{02}}{a_{20}\frac{A_n}{C_n} + a_{21}\frac{B_n}{C_n} + a_{22}},$$

i.e.,

$$\lim_{n\to\infty} \frac{\tilde{A}_n}{\tilde{C}_n} = \frac{a_{00}\sqrt[3]{d^2} + a_{01}\sqrt[3]{d} + a_{02}}{a_{20}\sqrt[3]{d^2} + a_{21}\sqrt[3]{d} + a_{22}} \tag{3.7}$$

and similarly

$$\lim_{n\to\infty}\frac{\tilde{B}_n}{\tilde{C}_n}=\frac{a_{10}\sqrt[3]{d^2}+a_{11}\sqrt[3]{d}+a_{12}}{a_{20}\sqrt[3]{d^2}+a_{21}\sqrt[3]{d}+a_{22}}. \tag{3.8}$$

Therefore, starting from the approximations of the previous theorem we can construct rational approximations, connected to generalized Rédei polynomials, for all these cubic irrationalities. However, we do not know if these approximations correspond to convergents of some bifurcating continued fraction. To make this happen, it is necessary to express the matrix

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

as a product of matrices of type (2.5). The problem of the factorization of any matrix into a product of the kind (2.5) is very difficult and we do not study it in this paper. However, we can obtain another interesting result. If we consider the product

$$A=\begin{pmatrix} a_0 & 1 & 0 \\ b_0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\cdots\begin{pmatrix} a_3 & 1 & 0 \\ b_3 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

the matrix A has first row with entries

$$a_0+a_3+a_0a_1a_2a_3+a_2a_3b_1+a_0a_3b_2+a_0a_1b_3+b_1b_3,\quad 1+a_0a_1a_2+a_2b_1+a_0b_2,\quad a_0a_1+b_1$$

and third row with entries

$$1+a_1a_2a_3+a_3b_2+a_1b_3,\quad a_1a_2+b_2,\quad a_1.$$

If these entries are matched with the entries of a generic matrix

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{pmatrix}$$

the system has rational solutions

$$\begin{cases} a_0=1,\quad a_1=a_{22} \\ a_2=\dfrac{1-a_{01}+a_{21}}{a_{22}-a_{02}},\quad a_3=\dfrac{a_{22}-a_{00}a_{22}+a_{02}a_{20}-a_{22}}{a_{02}a_{21}-a_{01}a_{22}} \\ b_0=1,\quad b_1=a_{02}-a_{22} \\ b_2=\dfrac{a_{01}a_{22}-a_{22}-a_{02}a_{21}}{a_{22}-a_{02}},\quad b_3=\dfrac{a_{21}-a_{00}a_{21}+a_{01}a_{20}-a_{01}}{a_{01}a_{22}-a_{02}a_{21}}. \end{cases}$$

Using these choices for a_i, b_i, $i=0,1,2,3$, the second row of the matrix A cannot clearly be any row, but it will be determined by these values. Therefore, we are able to construct a periodic bifurcating continued fraction for any cubic irrationality (3.7) paired with another determined cubic irrationality, starting from the fraction (3.5).

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