REVERSING DUCCI SEQUENCES

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ABSTRACT. Most of the work published on Ducci sequences is concerned with finding the behavior of the iterates of the Ducci map. Here we are interested in roughly the opposite. More precisely if T is the Ducci map and $\vec{x}_0 \in \mathbb{N}^k$ for some $k \in \mathbb{N}$, we seek $\vec{x}_1$ such that $T(\vec{x}_1) = \vec{x}_0$ and more generally a sequence $(x_n)_{n\in\mathbb{N}}$ of distinct vectors such that $T(\vec{x}_n) = \vec{x}_{n-1}$ for every $n \geq 1$. We prove that when k is odd, the existence of $\vec{x}_1$ implies the existence of $(x_n)_{n\in\mathbb{N}}$ and when $k = 2^l$ no vector in $\mathbb{N}^k$ has this property. Some other related results are deduced.

1. Introduction and Notation

Let $k \in \mathbb{N}$ and let $\vec{x} = (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{N}^k$. We define a map $T : \mathbb{N}^k \to \mathbb{N}^k$ by

$$T(\vec{x}) = T(a_0, a_1, \ldots, a_{k-1}) = (|a_0 - a_1|, |a_1 - a_2|, \ldots, |a_{k-1} - a_0|).$$

We call the map T the *Ducci map*. The sequence $(T^n(\vec{x}))_{n\in\mathbb{N}}$ of the iterations of T is called the *Ducci sequence* generated by $\vec{x}$. Ducci sequences were first discovered by Enrico Ducci and have been rediscovered by many authors since. The reference section lists some of many interesting articles on the subject.

In order to simplify the notation, the indices of the components of any vector $\vec{x} \in \mathbb{N}^k$ will be written modulo k so that, for example, $a_k = a_0$ and $a_{k+1} = a_1$. Note that we could define the map T over $\mathbb{Z}^k$, $\mathbb{Q}^k$ or $\mathbb{R}^k$ in exactly the same way. In this paper, we consider T over $\mathbb{N}^k$. However we will encounter in the proofs vectors with components in $\mathbb{Z}^k$ and $\mathbb{Q}^k$. Whenever this is the case we will take T to be the natural extension over $\mathbb{Z}^k$ or $\mathbb{Q}^k$.

2. Ancestors

For every $n \in \mathbb{N}$, we say that a vector $\vec{y}_n \in \mathbb{N}^k$ is an *n-ancestor* of the vector $\vec{x} \in \mathbb{N}^k$ if $T^n(\vec{y}_n) = \vec{x}$ and all the $\vec{T}^i(y_n), 0 \leq i \leq n$, are distinct. A 1-ancestor will simply be called an *ancestor*. The next result gives a simple characterization of the existence of an ancestor. It was first proved in [9].

Lemma 2.1. *Fix $k \in \mathbb{N}$ and $\vec{x} \in \mathbb{N}^k$. The vector $\vec{x} = \{a_0, a_1, \ldots, a_{k-1}\}$ has an ancestor if and only if there exist $\epsilon_0, \epsilon_1, \ldots, \epsilon_{k-1} \in \{-1, +1\}$ such that:*

$$\sum_{i=0}^{k-1} \epsilon_i a_i = 0.$$

Proof. Suppose $(b_0, b_1, \ldots, b_{k-1})$ is an ancestor of $(a_0, a_1, \ldots, a_{k-1})$. Then for every $0 \leq i \leq k-1$, $a_i = |b_i - b_{i+1}|$ and we choose $\epsilon_i \in \{-1, 1\}$ such that $\epsilon_i a_i = b_i - b_{i+1}$. For this choice,

$$\sum_{i=0}^{k-1} \epsilon_i a_i = \sum_{i=0}^{k-1} b_i - b_{i+1} = 0.$$

Conversely suppose $\vec{x} = (a_0, a_1, \ldots, a_{k-1})$ is in $\mathbb{N}^k$ and $\Sigma_{i=0}^{k-1}\epsilon_i a_i = 0$ for some $\epsilon_i \in \{-1, 1\}$, $0 \leq i \leq k-1$. For every $0 \leq n \leq k-1$ define $b_n = \sum_{i=0}^{n} \epsilon_i a_i$.

Note that

$$\begin{aligned} T(b_{k-1}, b_0, b_1, \ldots, b_{k-2}) &= (|b_{k-1} - b_0|, |b_0 - b_1|, \ldots, |b_{k-2} - b_{k-1}|) \\ &= (|\epsilon_0 a_0|, |\epsilon_1 a_1|, \ldots, |\epsilon_{k-1} a_{k-1}|) \\ &= (a_0, a_1, \ldots, a_{k-1}). \end{aligned}$$

However the vector $\vec{y} = (b_{k-1}, b_0, \ldots, b_{k-2})$ does not necessarily belongs to $\mathbb{N}^k$ as it may have negative components. To correct that, let $M = max\{|b_0|, |b_1|, \ldots, |b_{k-1}|\}$ and define $\vec{y'} = (b_{k-1} + M, b_0 + M, \ldots, b_{k-2} + M)$. We now have $\vec{y'} \in \mathbb{N}^k$ and

$$\begin{aligned} T(\vec{y'}) &= T(b_{k-1} + M, b_0 + M, \ldots, b_{k-2} + M) \\ &= (|b_{k-1} - b_0|, |b_0 - b_1|, \ldots, |b_{k-2} - b_{k-1}|) \\ &= (|\epsilon_0 a_0|, |\epsilon_1 a_1|, \ldots, |\epsilon_{k-1} a_{k-1}|) = (a_0, a_1, \ldots, a_{k-1}). \end{aligned}$$

□

A vector with no ancestor will be called an *original* vector. A vector with an n-ancestor for every $n \in \mathbb{N}$ will be said to be *ageless*. A sequence $(\vec{x}_n)_{n \leq N}$ of distinct vectors such that $\vec{x} = \vec{x}_0$ and $T(\vec{x}_n) = \vec{x}_{n-1}$ for every $0 < n \leq N$ will be called an *N-ascendance* of $\vec{x}$. A sequence $(\vec{x}_n)_{n \in \mathbb{N}}$ of distinct vectors such that $\vec{x} = \vec{x}_0$ and $T(\vec{x}_n) = \vec{x}_{n-1}$ for every $n > 0$ will be called an *∞-ascendance* of $\vec{x}$. A vector $\vec{x}$ which admits an ∞-ascendance will be called *eternal*. Let $\vec{x} = (a_0, a_1, \ldots a_{k-1}) \in \mathbb{N}^k$. If there exists $a \in \mathbb{N}$ such that for every $0 \leq i \leq k$, $a_i \in \{0, a\}$, we will say that $\vec{x}$ is a *simple* vector. The following is a well-known result (see for example [6]).

Theorem 2.2. *For every integer k and every $\vec{x} \in \mathbb{N}^k$, there exists $n \in \mathbb{N}$ such that $T^n(\vec{x})$ is simple.*

3. When k is Odd

Proposition 3.1. *Let $k \in \mathbb{N}$. If k is odd then every nonzero $\vec{x} \in \mathbb{N}^k$ is either original or eternal.*

Proof. Let $k \in \mathbb{N}$ be an odd number and $\vec{x}_0 \in \mathbb{N}^k$ a nonzero vector which is not original. Our goal is to construct inductively a sequence of distinct vectors $(\vec{x}_n)_{n \in \mathbb{N}}$ in $\mathbb{N}^k$ such that $T(\vec{x}_n) = \vec{x}_{n-1}$ for every $n \geq 1$.

When $\vec{x}_0$ is not a simple vector, it is sufficient to construct an ancestor $\vec{x}_1$ of $\vec{x}_0$ which itself is not original and iterate the process. Indeed since only a simple vector can belong to a cycle, we are guaranteed that all the $\vec{x}_n$ constructed this way will be distinct. We finish the proof in this case and will return to the simple vector case later on.

Let $\vec{y}_1$ be any ancestor of $\vec{x}_0$. If $\vec{y}_1$ is not original, we simply set $\vec{x}_1 = \vec{y}_1$. In case $\vec{y}_1 = (b_0, b_1, \ldots, b_{k-1})$ does not have an ancestor we proceed as follows. Let $m = min\{b_i : 0 \leq i \leq k-1\}$. The vector $\vec{z}_1 = (b_0 - m, b_1 - m, \ldots, b_{k-1} - m) = (a_0, a_1, \ldots, a_{k-1})$ is also an ancestor of $\vec{x}_0$ and at least one of its coordinates is 0. In order to simplify the notation we will assume that $a_0 = 0$. The reader can check that the proof below can easily be adapted to the other cases.

In case $\vec{z}_1$ has itself an ancestor, we can again simply set $\vec{x}_1 = \vec{z}_1$. If $\vec{z}_1$ does not have an ancestor Lemma 2.1 implies

$$\sum_{i=0}^{\frac{k-1}{2}} a_i \neq \sum_{i=\frac{k+1}{2}}^{k-1} a_i$$

and since $a_0 = 0$ this means either

$$\sum_{i=1}^{\frac{k-1}{2}} a_i > \sum_{i=\frac{k+1}{2}}^{k-1} a_i \qquad \text{or} \qquad \sum_{i=1}^{\frac{k-1}{2}} a_i < \sum_{i=\frac{k+1}{2}}^{k-1} a_i.$$

In the first case we define $d = \sum_{i=1}^{\frac{k-1}{2}} a_i - \sum_{i=\frac{k+1}{2}}^{k-1} a_i > 0$. Using the fact that $a_0 = 0$ we obtain:

$$\left(\sum_{i=\frac{k+1}{2}}^{k-1} a_i\right) + d = \sum_{i=1}^{\frac{k-1}{2}} a_i$$

$$\left(\sum_{i=\frac{k+1}{2}}^{k-1} (a_i + d)\right) + (a_0 + d) = \sum_{i=1}^{\frac{k-1}{2}} (a_i + d).$$

The last equality implies

$$\sum_{i=0}^{k-1} \epsilon_i (a_i + d) = 0$$

for the choice of $\epsilon_i = 1$ if $\frac{k+1}{2} \leq i \leq k-1$ or $i = 0$ and $\epsilon_i = -1$ if $1 \leq i \leq \frac{k-1}{2}$. By Lemma 2.1 the vector $(a_0 + d, a_1 + d, \ldots, a_{k-1} + d)$ has an ancestor. Since we also have $T(a_0 + d, a_1 + d, \ldots, a_{k-1} + d) = \vec{x}_0$, we can set $\vec{x}_1 = (a_0 + d, a_1 + d, \ldots, a_{k-1} + d)$.

The second case is similar. We define $d = \sum_{i=\frac{k+1}{2}}^{k-1} a_i - \sum_{i=1}^{\frac{k-1}{2}} a_i > 0$. Using the fact that $a_0 = 0$ we obtain:

$$\left(\sum_{i=1}^{\frac{k-1}{2}} a_i\right) + d = \sum_{i=\frac{k+1}{2}}^{k-1} a_i$$

$$\left(\sum_{i=1}^{\frac{k-1}{2}} (a_i + d)\right) + (a_0 + d) = \sum_{i=\frac{k+1}{2}}^{k-1} (a_i + d).$$

The last equality implies

$$\sum_{i=0}^{k-1} \epsilon_i (a_i + d) = 0$$

for the choice of $\epsilon_i = 1$ if $0 \leq i \leq \frac{k-1}{2}$ and $\epsilon_i = -1$ if $\frac{k+1}{2} \leq i \leq k-1$. By Lemma 2.1 the vector $(a_0+d, a_1+d, \ldots, a_{k-1}+d)$ has an ancestor. Since it also satisfies $T(a_0+d, a_1+d, \ldots, a_{k-1}+d) = \vec{x}_0$, we set $\vec{x}_1 = (a_0 + d, a_1 + d, \ldots, a_{k-1} + d)$ and the proof is complete when $\vec{x}_0$ is not a simple vector.

We now consider the case when $\vec{x}_0$ is a simple vector. First notice that we can assume that the components of $\vec{x}_0$ are in $\{0, 1\}$. Indeed if the components of $\vec{x}_0$ are in $\{0, a\}$ we define

$\vec{x}'_0 = \frac{1}{a} \cdot \vec{x}_0$. The components of $\vec{x}'_0$ are all in $\{0,1\}$ and if we can find a sequence $\vec{x}'_n$ satisfying $T^n(\vec{x}'_n) = \vec{x}'_0$ then the sequence $a \cdot \vec{x}_n$ satisfies $T^n(a \cdot \vec{x}'_n) = a \cdot \vec{x}'_0 = \vec{x}_0$.

Let $\vec{x}_0 = (a_0, a_1, \dots, a_{k-1})$ be a simple vector with components in $\{0,1\}$. Our goal is to find an ancestor of $\vec{x}_0$ which is neither simple nor original. Since we assumed $\vec{x}_0$ is not original, Lemma 2.1 implies it cannot be $(1,1,\dots,1)$ and since we assumed it is a nonzero vector it cannot be $(0,0,\dots,0)$. In particular, there must be an index j for which $a_j = 1$ and $a_{j+1} = 0$. Consider now the vector $\vec{X}_0$ obtained from $\vec{x}_0$ by replacing a_{j+1} by 2, i.e, $\vec{X}_0 = (A_0, A_1, \dots, A_{k-1})$ where $A_i = a_i$ if $i \neq j+1$ and $A_i = a_i + 2$ if $i = j+1$. For every $0 \leq i \leq k-1$ we have $A_i \equiv a_i \pmod 2$ and it follows easily that for every $n \in \mathbb{N}$,

$$T^n(\vec{x}_0) = T^n(\vec{X}_0) \pmod 2. \tag{3.1}$$

By Theorem 2.2, there exists $n_0 \in \mathbb{N}$ such that the vector $T^{n_0}(\vec{X}_0)$ is simple. We claim that the components of $T^{n_0}(\vec{X}_0)$ are in $\{0,1\}$. Indeed, up to a rotation the vector $\vec{X}_0$ is of the following form: (with possibly no 0 between 2 and the next 1)

$$(\dots, 1, 2, 0, \dots, 0, 1 \dots).$$

Note that the 2 will not propagate due to the presence of a 1 to its left. It is also easy to see that the "gap" between the unique 2 and next closest 1 on its right will only decrease and that eventually the 2 will disappear, leaving only 1 and 0's.

Using (3.1), we obtain that the vector $\vec{X}_0$ is a non-simple vector which eventually reaches the cycle in which $\vec{x}_0$ belongs. It only remains to show that $\vec{X}_0$ is not original itself. Since $\vec{x}_0$ is not original, Theorem 2.1 implies that it has an even number of 1's. Since it is not $\vec{0}$, the number of 1's is even and at least two. In particular $\vec{X}_0$ satisfies the condition of Theorem 2.1 and therefore is not original itself, concluding the proof. □

Combining Lemma 2.1 and Proposition 3.1 we obtain the following corollary.

Corollary 3.2. *Given $k \in \mathbb{N}$ odd and $\vec{x} = (a_0, a_1, \dots, a_{k-1}) \in \mathbb{N}^k$, the vector $\vec{x}$ is eternal if and only if there exist $\epsilon_0, \epsilon_1, \dots, \epsilon_{k-1} \in \{-1, +1\}$ such that:*

$$\sum_{i=0}^{k-1} \epsilon_i a_i = 0.$$

Otherwise $\vec{x}$ is original.

The proof of Proposition 3.1 was based on finding an ancestor for $\vec{x}$ which has an ancestor itself and iterating the process. As it turns out, at any step in this process we could have picked an ancestor which is itself original, thus showing the following result.

Corollary 3.3. *If k is odd and $n \in \mathbb{N}$, any eternal vector in $\mathbb{N}^k$ has an n-ancestor which is original.*

Proof. Let k and $\vec{x}$ be as in the statement of the Corollary. By the Proposition 3.1, there exists an n-ancestor $\vec{y} = (y_0, y_1, \dots, y_{k-1})$ of $\vec{x}$. Set

$$L = \sum_{i=0}^{k-1} y_i$$

and define $\vec{y'} = (y_0 + L, y_1 + L, \dots, y_{k-1} + L)$. Clearly $T^n(\vec{y'}) = \vec{x}$ and we claim that $\vec{y'}$ has no ancestor. If $\vec{y'}$ has an ancestor, then by Lemma 2.1 there exists $\epsilon_i \in \{-1, 1\}$, $0 \leq i \leq k-1$

such that

$$\sum_{i=0}^{k-1} \epsilon_i(y_i + L) = 0.$$

Consequently,

$$0 = \left|\sum_{i=0}^{k-1} \epsilon_i(y_i + L)\right| = \left|\sum_{i=0}^{k-1} \epsilon_i y_i + \sum_{i=0}^{k-1} \epsilon_i L\right| \geq \left| |\sum_{i=0}^{k-1} \epsilon_i L| - |\sum_{i=0}^{k-1} \epsilon_i y_i| \right|. \tag{3.2}$$

However, since k is odd we also have

$$\left|\sum_{i=0}^{k-1} \epsilon_i L\right| \geq |L| > \left|\sum_{i=0}^{k-1} \epsilon_i y_i\right|,$$

making the last term of (3.2) positive, a contradiction. □

As a final remark, note that we could replace L in the previous proof by $L + m$ for any $m \in \mathbb{N}$. Consequently we can strengthen the previous result.

Corollary 3.4. *If k is odd and $n \in \mathbb{N}$, then any eternal vector in $\mathbb{N}^k$ has infinitely many n-ancestors which are original.*

4. When k is a Power of Two

If there exists $n \in \mathbb{N}$ such that $T^n(\vec{x}) = \vec{0}$, we say that $\vec{x}$ is nilpotent. We will use the following well-known result (see for example [8, 1] or [2] for an extension over $\mathbb{Z}_2$).

Theorem 4.1. *If $k = 2^l$ for some $l \in \mathbb{N}$, then every vector in $\mathbb{N}^k$ is nilpotent.*

Theorem 4.1 implies the following useful lemma. A strengthening of this lemma can be found in [1], where the authors showed that all the components of $T^k(\vec{x})$ are even.

Lemma 4.2. *For every $l \in \mathbb{N}$ if $k = 2^l$ and $\vec{x} \in \mathbb{N}^k$, the components of $T^{2^k}(\vec{x})$ are all even.*

Proof. For any $\vec{x} = (a_0, a_1, \ldots, a_{k-1}) \in \mathbb{N}^k$, consider $e(\vec{x}) = (b_0, b_1, \ldots, b_{k-1}) \in \{1, 0\}^k$ defined by $b_i = a_i \pmod 2$. It is easy to see that

$$e(T(\vec{x})) = T(e(\vec{x})). \tag{4.1}$$

Since there are only 2^k vectors with components in $\{0, 1\}$, Theorem 4.1 implies that $T^{2^k}(e(\vec{x})) = \vec{0}$. Using 4.1, we obtain $e(T^{2^k}(\vec{x})) = \vec{0}$ which implies that the components of $T^{2^k}(\vec{x})$ are all even. □

The next proposition shows that the case $k = 2^l$ differs dramatically from the case when k is odd.

Proposition 4.3. *Let $l \geq 2$ and $k = 2^l$. No vector in $\mathbb{N}^k$ is eternal and the only ageless vector is $\vec{0}$.*

Proof. Let $k = 2^l$ for some integer $l \geq 2$ and suppose $\vec{x}_0 \neq \vec{0}$ is ageless or eternal. Define m to be the largest integer such that 2^m divides each of the components of $\vec{x}_0$ and $N = 2^k \cdot (m+1)$. Consider $(\vec{x}_n)_{n \leq N}$ an N-ascendance of $\vec{x}_0$.

By Lemma 4.2, each of the $\vec{x}_n$ for $n \leq N - 2^k$ must have all their components even. It is easy to check that $(\vec{x}_n/2)_{n \leq N-2^k}$ is an $(N - 2^k)$-ascendance for $\vec{x}_0/2$ and all components of $\vec{x}_n/2$, $n \leq N - 2^k$, are integers. We can repeat the argument with $\vec{x}_0/2$ and $(\vec{x}_n/2)_{n \leq N-2^k}$ to show that all the components of $\vec{x}_n/4$, $n \le N - 2 \cdot 2^k$, are integers. Iterating the process $m+1$ times shows that the components of $\vec{x}_0/2^{m+1}$ are all integers, in contradiction with our choice of m.

Therefore if an eternal or ageless vector exists, it has to be $\vec{0}$. However if $(x_n)_{n\in\mathbb{N}}$ is an ∞-ascendance of $\vec{0}$, $\vec{x}_1$ is a nonzero eternal vector, contradicting the result from the previous paragraph. Consequently, $\vec{0}$ is not eternal.

It remains to show that $\vec{0}$ is ageless. We will prove it for $k = 4$. The general statement follows by concatenation of vectors of length 4. By Theorem 4.1, it is sufficient to show that there exists for every $n \ge 0$ a vector $\vec{x} \in \mathbb{N}^4$ such that $T^n(\vec{x}_n) \ne \vec{0}$. This follows directly from [12] or [3] but we include here a different proof.

Let $a, b, c \in \mathbb{N}$ satisfying the following conditions:

(1) $a \ne 0$.
(2) $a < b < c$.
(3) $a + b < c$.

Let $\vec{v}_0 = (0, a, b, c)$ and $\theta = \frac{c-a-b}{2}$. By (3) the constant θ is positive. Note that

$$\begin{aligned}
|a + 2\theta - (c + \theta)| &= |a + \theta - c| = |a + \frac{c-a-b}{2} - c| \\
&= |a/2 - b/2 - c/2| = |-b + a/2 + b/2 - c/2| \\
&= |-b - \theta| = b + \theta.
\end{aligned}$$

This implies

$$T(0, \theta, a + 2\theta, c + \theta) = (\theta, a + \theta, b + \theta, c + \theta).$$

By construction $T^2(0, \theta, a + 2\theta, c + \theta) = T(0, a, b, c) \ne (0, 0, 0, 0)$. Notice that

(1) $\theta \ne 0$.
(2) $\theta < a + 2\theta < c + \theta$.
(3) $\theta + 2\theta + a < c + \theta$.

Consequently the vector $\vec{v}_1 = (0, \theta, a + 2\theta, c + \theta) \in \mathbb{Q}^4$ satisfies the same three conditions as $\vec{v}_0$ and the process can be iterated. This allows us to construct a sequence $\vec{v}_n \in \mathbb{Q}^4$ satisfying $T^n(\vec{v}_n) \ne \vec{0}$. To complete the proof, define d to be the least common denominator of the fractions occurring as components of $\vec{v}_n$. Define $\vec{x}_n = d \cdot \vec{v}_n$, where the multiplication is component wise. The vector $\vec{x}_n$ lies in $\mathbb{N}^4$ and satisfies $T^n(\vec{v}_n) \ne \vec{0}$, concluding the proof. $\square$

5. Conclusion and Future Directions

Beside Lemma 2.1 which is true for any k, the results of this paper can be adapted to a certain extent to k even. Consider for example the vector $\vec{x} = (1, 3, 2, 1, 3, 2)$ in $\mathbb{N}^6$. It is the concatenation of two copies of $\vec{y} = (1, 3, 2)$ and it is easy to see that if (a_0, a_1, a_2) is an n-ancestor of $\vec{y}$, then $(a_0, a_1, a_2, a_0, a_1, a_2)$ is an n-ancestor of $\vec{x}$. Since $1 - 3 + 2 = 0$, Corollary 3.2 shows that $\vec{y}$, and thus $\vec{x}$ is eternal. This simple example can be generalized to show that for any k which is not a power of 2, there are eternal vectors in $\mathbb{N}^k$.

In general it would be interesting to determine what is the "oldest" ancestor of a given vector $\vec{x}$. More precisely consider the following question.

Question 1. *Given $\vec{x} \in \mathbb{N}^k$, if it exists, what is the largest integer n such that there exist $\vec{x}_n$ satisfying $T^n(\vec{x}_n) = \vec{x}$? If such an integer does not exist, is $\vec{x}$ ageless or eternal?*

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