CONGRUENT NUMBERS AND CONTINUED FRACTIONS

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Abstract. We discuss the continued fraction expansion $[a_0; \overline{a, b}]$, whose limit is related to rational right triangles of area close to some positive number. Our results include those of Steuding [3].

1. Introduction

A *congruent number* is a positive integer n for which there exists a right triangle having area n and rational sides. The sequence of congruent numbers starts with

$$5, 6, 7, 13, 14, 15, 20, 21, 22, 23, 24, 28, 29, 30, 31, 34, 37, 38, 39, 41, 45, 46, 47, 52, 53, \dots$$

[2, A003273].

It is known that n is a congruent number if and only if the elliptic curve $E_n : y^2 = x^3 - n^2 x$ contains a rational point (x, y) with $y \neq 0$, equivalently, that the Mordell-Weil group $E_n(\mathbb{Q})$ of rational points has positive rank.

Recently, Steuding [3] studied a convergent sequence of Fibonacci ratios and showed that its limit is related to rational right triangles of area as close to one as one may like, by using the continued fraction expansion of $\sqrt{5} = [2; \overline{4}]$. Note that the indices in [3] are more or less wrong from the very beginning. In all results one needs to shift from n to $n+1$. For example, the first identity in [3, Theorem 2.1] should be $r_n = (F_{3n+4} + F_{3n+2})/F_{3n+3}$.

In this article we discuss the continued fraction expansion $[a_0; \overline{a, b}]$, whose limit is related to rational right triangles of area close to some positive number.

2. Continued Fraction of the Type $[a_0; \overline{a, b}]$

Consider the continued fractions expansion of

$$a_0 + \frac{1-\beta}{a} = a_0 + \frac{\sqrt{a^2b^2 + 4ab} - ab}{2a} = [a_0; \overline{a, b}],$$

where a and b are positive integers. Then we obtain the following theorem.

Theorem 2.1. *Let r_n $(n \geq 0)$ be the nth convergent of the continued fraction expansion of*

$$a_0 + \frac{\sqrt{a^2b^2 + 4ab} - ab}{2a}.$$

Then for $n \geq 0$

$$r_{2n} = a_0 + \frac{b}{\alpha - 1} + \mathcal{O}(\alpha^{-2n-2})$$

and

$$r_{2n+1} = a_0 + \frac{1-\beta}{a} + \mathcal{O}(\alpha^{-2n-2}).$$

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Proof. There are the recurrence formulas for $n \geq 1$

$$\begin{aligned} p_{2n-1} &= ap_{2n-2} + p_{2n-3} \\ p_{2n} &= bp_{2n-1} + p_{2n-2} \\ q_{2n-1} &= aq_{2n-2} + q_{2n-3} \\ q_{2n} &= bq_{2n-1} + q_{2n-2} \end{aligned}$$

with $p_0 = a_0$, $p_{-1} = 1$, $p_{-2} = 0$, $q_0 = 1$, $q_{-1} = 0$, and $q_{-2} = 1$. Hence, using

$$p_{2n} = (ab+2)p_{2n-2} - p_{2n-4}$$

we obtain

$$\begin{aligned} p_{2n} &= \frac{a_0(\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta} \\ &= \frac{a_0(\alpha^{n+1} - \beta^{n+1}) + (b - a_0)(\alpha^n - \beta^n)}{\alpha - \beta} \quad (n \geq 0), \end{aligned}$$

where

$$\alpha = \frac{ab + 2 + \sqrt{a^2b^2 + 4ab}}{2} \quad \text{and} \quad \beta = \frac{ab + 2 - \sqrt{a^2b^2 + 4ab}}{2}$$

with $\alpha + \beta = ab + 2$ and $\alpha\beta = 1$. Similarly, using

$$p_{2n+1} = (ab+2)p_{2n-1} - p_{2n-3}$$

we obtain

$$p_{2n+1} = \frac{(a_0a + 1)(\alpha^{n+1} - \beta^{n+1}) - (\alpha^n - \beta^n)}{\alpha - \beta} \quad (n \geq 0).$$

By $q_{2n} = (ab+2)q_{2n-2} - q_{2n-4}$ we have

$$q_{2n} = \frac{(\alpha^{n+1} - \beta^{n+1}) - (\alpha^n - \beta^n)}{\alpha - \beta} \quad (n \geq 0).$$

By $q_{2n+1} = (ab+2)q_{2n-1} - q_{2n-3}$ we have

$$q_{2n+1} = \frac{a(\alpha^{n+1} - \beta^{n+1})}{\alpha - \beta} \quad (n \geq 0).$$

Therefore,

$$\begin{aligned} r_{2n} = \frac{p_{2n}}{q_{2n}} &= a_0 + \frac{b(\alpha^n - \beta^n)}{(\alpha^{n+1} - \beta^{n+1}) - (\alpha^n - \beta^n)} \\ &= a_0 + \frac{b}{\alpha - 1} + \mathcal{O}(\alpha^{-2n-2}) \end{aligned}$$

and

$$\begin{aligned} r_{2n+1} = \frac{p_{2n+1}}{q_{2n+1}} &= a_0 + \frac{1}{a} - \frac{\alpha^n - \beta^n}{a(\alpha^{n+1} - \beta^{n+1})} \\ &= a_0 + \frac{1 - \beta}{a} + \mathcal{O}(\alpha^{-2n-2}). \end{aligned}$$

□

Theorem 2.2. *Let*

$$a_n = \frac{3}{2r_n}, \quad b_n = \frac{20}{3r_n}, \quad c_n = \frac{41}{6r_n} \quad (n \geq 0).$$

Then the triple a_n*,* b_n*, and* c_n *are the lengths of the sides of a rational right triangle of area*

$$\frac{5}{(a_0 + b/(\alpha - 1))^2} + \mathcal{O}(\alpha^{-2n-2})$$

or

$$\frac{5}{(a_0 + (1-\beta)/a)^2} + \mathcal{O}(\alpha^{-2n-2}).$$

Proof.

$$S_{2n} = \frac{1}{2}a_{2n}b_{2n} = \frac{5}{r_{2n}^2} = \frac{5}{(a_0 + \frac{b}{\alpha-1} + \mathcal{O}(\alpha^{-2n-2}))^2}$$
$$= \frac{5}{(a_0 + b/(\alpha - 1))^2} + \mathcal{O}(\alpha^{-2n-2})$$

or

$$S_{2n+1} = \frac{1}{2}a_{2n+1}b_{2n+1} = \frac{5}{r_{2n+1}^2} = \frac{5}{(a_0 + \frac{1}{\alpha} - \frac{\beta}{a} + \mathcal{O}(\alpha^{-2n-2}))^2}$$
$$= \frac{5}{(a_0 + (1-\beta)/a)^2} + \mathcal{O}(\alpha^{-2n-2}).$$

□

Similarly, we obtain the following theorem.

Theorem 2.3. *Let*

$$a_n = \frac{3}{r_n}, \quad b_n = \frac{4}{r_n}, \quad c_n = \frac{5}{r_n} \quad (n \geq 0).$$

Then the triple a_n*,* b_n*, and* c_n *are the lengths of the sides of a rational right triangle of area*

$$\frac{6}{(a_0 + b/(\alpha - 1))^2} + \mathcal{O}(\alpha^{-2n-2})$$

or

$$\frac{6}{(a_0 + (1-\beta)/a)^2} + \mathcal{O}(\alpha^{-2n-2}).$$

Proof.

$$S_{2n} = \frac{1}{2}a_{2n}b_{2n} = \frac{6}{r_{2n}^2} = \frac{6}{(a_0 + b/(\alpha - 1))^2} + \mathcal{O}(\alpha^{-2n-2})$$

or

$$S_{2n+1} = \frac{1}{2}a_{2n+1}b_{2n+1} = \frac{6}{r_{2n+1}^2} = \frac{6}{(a_0 + (1-\beta)/a)^2} + \mathcal{O}(\alpha^{-2n-2}).$$

□

For example, consider the irrational number $\sqrt{a^2+1} = [a; \overline{2a}]$. Since a is replaced by $2a$, $a_0 = a$ and $b = 2a$, we obtain

$$r_n = \frac{p_n}{q_n} = a + \frac{1}{\sqrt{a^2+1} + a} + \mathcal{O}(\alpha^{-2n-2}) = \sqrt{a^2+1} + \mathcal{O}(\alpha^{-2n-2}).$$

Hence, letting

$$a_n = \frac{3}{2r_n}, \quad b_n = \frac{20}{3r_n}, \quad c_n = \frac{41}{6r_n}$$

yields

$$S_n = \frac{1}{2}a_n b_n = \frac{5}{r_n^2} = \frac{5}{a^2+1} + \mathcal{O}(\alpha^{-2n-2}).$$

If we let $a = 2$ in this case, then since $\phi^3 = ((\sqrt{5}+1)/2)^3 = \sqrt{5}+2 = \alpha$, we have

$$\frac{1}{2}a_n b_n = \frac{5}{r_n^2} = 1 + \mathcal{O}(\phi^{-6n-6}),$$

which is [3, Theorem 2.2]. Note that in his result one needs to shift from n to $n+1$.

In the next example, consider the irrational number $\sqrt{a^2+2} = [a; \overline{a, 2a}]$. Since $a_0 = a$ and $b = 2a$, we have

$$\begin{aligned} r_{2n} = \frac{p_{2n}}{q_{2n}} &= a + \frac{2a}{\alpha - 1} + \mathcal{O}(\alpha^{-2n-2}) \\ &= \sqrt{a^2+2} + \mathcal{O}(\alpha^{-2n-2}) \end{aligned}$$

and

$$\begin{aligned} r_{2n+1} = \frac{p_{2n+1}}{q_{2n+1}} &= a + \frac{1-\beta}{a} + \mathcal{O}(\alpha^{-2n-2}) \\ &= \sqrt{a^2+2} + \mathcal{O}(\alpha^{-2n-2}). \end{aligned}$$

Hence, letting

$$a_n = \frac{3}{2r_n}, \quad b_n = \frac{20}{3r_n}, \quad c_n = \frac{41}{6r_n}$$

yields

$$S_n = \frac{1}{2}a_n b_n = \frac{5}{r_n^2} = \frac{5}{a^2+2} + \mathcal{O}(\alpha^{-n-1}).$$

Setting

$$a_n = \frac{3}{r_n}, \quad b_n = \frac{4}{r_n}, \quad c_n = \frac{5}{r_n}$$

yields

$$S_n = \frac{1}{2}a_n b_n = \frac{6}{r_n^2} = \frac{6}{a^2+2} + \mathcal{O}(\alpha^{-n-1}).$$

3. A General Case

In general, consider the positive irrational number $\theta = [d_0; d_1, d_2, \dots]$, whose kth convergent is $r_k = p_k/q_k = [d_0; d_1, \dots, d_k]$. Choose a congruent number n, whose right triangle has the rational sides a, b, and c. The first ten congruent numbers with the side lengths of the associated right triangles are in the table (see e.g. [1]).

n	sides (a, b, c)
5	$\frac{3}{2}, \frac{20}{3}, \frac{41}{6}$
6	$3, 4, 5$
7	$\frac{24}{5}, \frac{35}{12}, \frac{337}{60}$
13	$\frac{780}{323}, \frac{323}{30}, \frac{106921}{9690}$
14	$\frac{8}{3}, \frac{21}{2}, \frac{65}{6}$
15	$\frac{15}{2}, 4, \frac{17}{2}$
20	$3, \frac{40}{3}, \frac{41}{3}$
21	$\frac{7}{2}, 12, \frac{25}{2}$
22	$\frac{33}{35}, \frac{140}{3}, \frac{4901}{105}$
23	$\frac{80155}{20748}, \frac{41496}{3485}, \frac{905141617}{72306780}$

If we let

$$a_k = \frac{a}{r_k}, \quad b_k = \frac{b}{r_k}, \quad c_k = \frac{c}{r_k}$$

then from the fact that $r_k \to \theta$ $(k \to \infty)$, the area of the right triangle is

$$S_k = \frac{1}{2} a_k b_k = \frac{n}{r_k^2} \to \frac{n}{\theta^2} \quad (k \to \infty).$$

References

[1] K. E. Morrison, *Congruent numbers*, http://www.aimath.org/news/congruentnumbers/congruentnumbers.pdf.

[2] OEIS Foundation Inc. (2011), The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[3] J. Steuding, *What Fibonacci numbers have to do with congruent numbers?*, The Fibonacci Quarterly, **49.4** (2011), 330–333.

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